HOMEWORK 4 - MATH 325
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Problem 1 *Use Figure 5.24 on page 75 of your textbook to show how to construct a pair of lengths* x, y *with given difference* $y - x = a$ *and given product* $xy = b^2$. *Make sure to prove that your construction works!*

Solution: On the sides of a right angle $\widehat{O}$, consider points A and B, respectively, such that $AO = \frac{a}{2}$ and $BO = b$. Draw the segment $\overline{AB}$ and with center A and radius $\overline{AB}$ draw a circle intersecting $\overline{AO}$ at C. Then $x = OC$.

We have, by construction and the pythagorean theorem $AB^2 = (\frac{a}{2})^2 + b^2$, whence $x = OC = AC - OA = AB - OA = -\frac{1}{2} + \sqrt{\frac{a^2}{4} + b^2}$. Thus x is the solution of the quadratic $x^2 + ax - b^2 = 0$, i.e., of $x(x + a) = b^2$, which is the given requirement for x. ■

Problem 2 *Use Figure 5.11 of your textbook to prove that for any lengths* a *and* b, $\sqrt{ab} \leq \frac{1}{2}(a+b)$. *This fact is sometimes called the* algebraic-geometric mean inequality.

Solution: $2x$ is the length of a chord of a circle with diameter $a + b$. therefore $2x \leq a + b$, i.e., $2\sqrt{ab} \leq a + b$ or $\sqrt{ab} \leq \frac{a+b}{2}$. ■

Problem 3 *Let* $\overline{AB}$ *and* $\overline{CD}$ *be parallel chords in a circle. Prove that* $\widehat{CAB} \cong \widehat{DBA}$.

Solution: Since $\overline{AB} \parallel \overline{CD}$, we have $\widehat{CAB} + \widehat{ACD} = 180^o$. But we also have $\widehat{ACD} + \widehat{DBA} = 180^o$, since they are opposite angles of an inscribed quadrilateral. Therefore $\widehat{CAB} = 180^o - \widehat{ACD} = 180^o - (180^o - \widehat{DBA}) = \widehat{DBA}$. ■

Problem 4 *Given a circle with center* O *and three tangent lines* l_1, l_2 *and* l_3 *such that* l_1 *and* l_2 *are parallel and* l_3 *intersects* l_1 *at* A *and* l_2 *at* B, *prove that* $\widehat{AOB} = 90^o$.

Solution: Let P, Q, R be the points of tangency of l_1, l_2, l_3, respectively with the given circle. Then we have

$$\begin{aligned} \widehat{AOB} &= 180^o - \widehat{RAO} - \widehat{RBO} \\ &= 180^o - \tfrac{1}{2}\widehat{RAP} - \tfrac{1}{2}\widehat{RBQ} \\ &= 180^o - \tfrac{1}{2}(\widehat{RAP} + \widehat{RBQ}) \\ &= 180^o - \tfrac{1}{2}180^o \\ &= 90^o. \end{aligned}$$

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Problem 5 *Given a line segment* $\overline{AB}$, *define* S *to be the set of points* C *on a given side of* $\overline{AB}$ *such that* $\widehat{ACB} = 45^o$. *What can you say about* S?

Solution: It is a point on the arc of the circle with center the unique point O of the perpendicular bisector of $\overline{AB}$, such that $\widehat{AOB} = 90^o$ and with radius $\overline{OA}$. To see this, recall that an angle subtending an arc is half the corresponding central angle. ■

Problem 6 *For which* n *satisfying* $3 \leq n \leq 25$ *is it possible to construct a regular* n*-gon?*

Solution: The numbers that can be expressed as a power of 2 times a product of distinct Fermat primes are

$$3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24.$$

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Problem 7 *For which of these n is a regular n-gon constructible?*
(a) 144 (b) 85 (c) 176 (d) 100 (e) 128.

Solution: (a) $144 = 2^4 3^2$ so 144-gon is not constructible.
(b) $85 = 5 \cdot 17$ so 85-gon is constructible.
(c) $176 = 2^4 \cdot 11$, whence 176-gon is not constructible.
(d) $100 = 2^2 5^2$, whence 100-gon is not constructible.
(e) $128 = 2^7$, whence 128-gon is constructible. ■

Problem 8 *What lower bounds do you get for C and A if you use a square inscribed in the circle?*

Solution: $C \geq 4\sqrt{2}R$ and $A \geq 2R^2$. ■

Problem 9 *What upper bounds do you get for C and A if you use a hexagon circumscribed about the circle?*

Solution: $C \leq 4\sqrt{3}R$ and $A \leq 3\sqrt{3}R^2$. ■