HOMEWORK 5 - MATH 325
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Problem 1 *Prove: if s is the length of a side of an n-gon inscribed in a circle of radius r and t the length of a side of a $2n$-gon inscribed in a circle of radius r, then $t^2 = 2r(r - \sqrt{r^2 - \frac{1}{4}s^2})$.*

Solution: Denote by x the distance on the radius from the middle of the side of the n-gon to the vertex of the $2n$-gon. Then we have

$$\begin{aligned} t^2 &= \tfrac{s^2}{4} + x^2 \\ &= \tfrac{s^2}{4} + (r - \sqrt{r^2 - \tfrac{s^2}{4}})^2 \\ &= \tfrac{s^2}{4} + r^2 + r^2 - \tfrac{s^2}{4} - 2r\sqrt{r^2 - \tfrac{s^2}{4}} \\ &= 2r^2 - 2r\sqrt{r^2 - \tfrac{s^2}{4}} \\ &= 2r(r - \sqrt{r^2 - \tfrac{s^2}{4}}). \end{aligned}$$

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Problem 2 *Use the previous exercise to get a lower bound for C based on the regular 24-gon; the regular 48-gon.*

Solution: In your book, there is an estimate using the length of the side of the 12-gon. That side has length $s_{12} = (\sqrt{2 - \sqrt{3}})r$. To compute the length of the side of the 24-gon, use the length of the side of the 12-gon and the formula of the previous problem. Let s_{24} be the resulting length. Then the lower bound for C would be $C \geq 24s_{24}$. Similarly, we may use s_{24} and the formula of the previous problem to obtain s_{48}. Then $C \geq 48s_{48}$. ■

Problem 3 *Assume that $\triangle ABC$ and $\triangle DEF$ are similar with ratio k. Prove that each of the circumradius, the inradius and the exradii of $\triangle DEF$ are k times the corresponding parts of $\triangle ABC$.*

Solution: Let O be the circumcenter of $\triangle ABC$ and A' the midpoint of $\overline{BC}$. Similarly, let P be the circumcenter of $\triangle DEF$ and D' the midpoint of $\overline{EF}$. Then, we have $\widehat{OA'B} = \widehat{PD'E} = 90^o$ and $\widehat{BOA'} \cong \widehat{A} \cong \widehat{D} \cong \widehat{EPD'}$, whence $\triangle BOA' \sim \triangle EPD'$, with ratio k. Thus, $OB = k \cdot PE$.

Next, let I be the inradius of $\triangle ABC$ and J be the inradius of $\triangle DEF$. Compare $\triangle ABI$ and $\triangle DEJ$. We have $\widehat{BAI} = \frac{\widehat{A}}{2} = \frac{\widehat{D}}{2} = \widehat{EDJ}$ and similarly $\widehat{ABI} = \widehat{DEJ}$. Therefore $\triangle ABI \sim \triangle DEJ$, whence $AI = k \cdot DJ$.

The exradii may be treated similarly with the inradii. ■

Problem 4 *In each part determine the missing information about $\triangle ABC$.*
(a) $a = 4, b = 4, c = 2, K = ?, R = ?, r = ?, r_a = ?, r_b = ?, r_c = ?$
(b) $r_a = 2, r_b = 3, r_c = 6, r = ?, K = ?, R = ?, a = ?, b = ?, c = ?$

Solution: (a) $s = \frac{a+b+c}{2} = \frac{4+4+2}{2} = 5$, whence

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{15},$$

$$R = \frac{abc}{4K} = \frac{4 \cdot 4 \cdot 2}{4\sqrt{15}} = \frac{8}{\sqrt{15}}, \quad r = \frac{K}{s} = \frac{\sqrt{15}}{5},$$

$$r_a = \frac{K}{s-a} = \sqrt{15}, \quad r_b = \frac{K}{s-b} = \sqrt{15}, \quad r_c = \frac{K}{s-c} = \frac{\sqrt{15}}{3}.$$

(b)

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1 \implies r = 1,$$

$$K = \sqrt{r r_a r_b r_c} = \sqrt{1 \cdot 2 \cdot 3 \cdot 6} = 6, \quad s = \frac{K}{r} = 6,$$

$$a = s - \frac{K}{r_a} = 3, \quad b = s - \frac{K}{r_b} = 4, \quad c = s - \frac{K}{r_c} = 5,$$

$$R = \frac{abc}{4K} = \frac{3 \cdot 4 \cdot 5}{4 \cdot 6} = \frac{5}{2}.$$

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Problem 5 *Prove that if a quadrilateral $ABCD$ is circumscribed about a circle, then the area of $ABCD$ is one-half times the radius of the circle times the perimeter.*

Solution: Let $ABCD$ be the circumscribed quadrilateral, E, F, G, H the point of tangency of $\overline{AB}, \overline{BC}, \overline{CD}, \overline{AD}$, respectively, with the circle and O the center of the circle. denote by R its radius and by p its perimeter. We have

$$\begin{aligned}
\text{Area}(ABCD) &= \text{Area}(AOB) + \text{Area}(BOC) + \text{Area}(COD) + \text{Area}(AOD) \\
&= \tfrac{1}{2}AB \cdot R + \tfrac{1}{2}BC \cdot R + \tfrac{1}{2}CD \cdot R + \tfrac{1}{2}AD \cdot R \\
&= \tfrac{1}{2}(AB + BC + CD + AD)R \\
&= \tfrac{1}{2}pR.
\end{aligned}$$

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Problem 6 *Let the lengths of the three altitudes be h_a, h_b and h_c. Prove that $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$.*

Solution:

$$\begin{aligned}
\tfrac{1}{r} &= \tfrac{s}{K} \\
&= \tfrac{a+b+c}{2K} \\
&= \tfrac{a}{2K} + \tfrac{b}{2K} + \tfrac{c}{2K} \\
&= \tfrac{1}{h_a} + \tfrac{1}{h_b} + \tfrac{1}{h_c}.
\end{aligned}$$

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Problem 7 *Prove that the circle with diameter $\overline{I_b I_c}$ has center on the circumcircle and contains the points B and C.*

Solution: Since $\widehat{I_cBI_b} = \widehat{I_cCI_b} = 90^o$, both B and C are points on the circle with diameter $\overline{I_bI_c}$. So it suffices to show that the point D of intersection of the circumcircle (O, R) of triangle $\triangle ABC$ with $\overline{I_bI_c}$ is the midpoint of $\overline{I_bI_c}$. Note that $\widehat{I_cDB} \cong \widehat{C}$, since they subtend the same arc AB. Therefore $\widehat{BDI_b} = 180^o - \widehat{I_cDB} = 180^o - \widehat{C} = \widehat{A} + \widehat{B}$. But note that I_bAIC are cocircular, whence $\widehat{DI_bB} \cong \widehat{ACI_c} = \frac{\widehat{C}}{2}$. Therefore, $\widehat{AI_bB} \cong \widehat{DBI_b} \cong \frac{\widehat{C}}{2}$. Thus, the triangle $\triangle DBI_b$ is isosceles and we get $\overline{DI_b} \cong \overline{DB}$. Similarly, we have that $\triangle DBI_c$ is also isosceles and we get $\overline{DI_c} \cong \overline{DB}$. Therefore $\overline{DI_b} \cong \overline{DI_c}$ and D is the midpoint of $\overline{I_bI_c}$. ■

Problem 8 *Prove* $OI_a^2 = R(R + 2r_a)$.

Solution: Follow the same reasoning that your book follows for proving Euler's Theorem in Section 7.6.

Let AI_a intersect the circumcircle at M, let P and Q be the points where I_aO intersects the circumcircle nd Z be the point of intersection of AB with the perpendicular from I_a to AB. Then we have $PI_a \cdot I_aQ = AI_a \cdot I_aM$ whence $(OI_a - R)(OI_a + R) = 2Rr_a$, since $\frac{AI_a}{r_a} = \frac{2R}{I_aM}$, by the similarity of $\triangle AZI_a \sim \triangle C'MC$, where C' is the antidiametric point of C, and the fact that $\overline{MC} \cong \overline{MI_a}$. Therefore $OI_a^2 = 2Rr_a + R^2 = R(2r_a + R)$. ■