HOMEWORK 1 - MATH 351
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Problem 1 *Prove each of the following using mathematical induction:*
(a) $2+5+8+\ldots+(3n-1)=\frac{n(3n+1)}{2}$.
(b) $1+t+t^2+\ldots+t^{n-1}=\frac{1-t^n}{1-t}$, *where* $t\neq 1$.
(c) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.
(d) $n^2-2n\geq n+3$ *for* $n\geq 4$.
(e) 7^n-2^n *is exactly divisible by* 5 *for* $n\geq 1$.

Solution: (a) For $n=1$, we get $2=\frac{1(3\cdot 1+1)}{2}$, which is true. Suppose that $2+5+\ldots+(3k-1)=\frac{n(3k+1)}{2}$. For $n=k+1$, we get

$$\begin{array}{rcl} 2+5+\ldots+(3k-1)+(3(k+1)-1) & = & \frac{k(3k+1)}{2}+3k+2 \\ & = & \frac{3k^2+k+6k+4}{2} \\ & = & \frac{3k^2+7k+4}{2} \\ & = & \frac{(k+1)(3k+4)}{2} \\ & = & \frac{(k+1)(3(k+1)+1)}{2}. \end{array}$$

(b) For $n=1$, we have $1=\frac{1-t}{1-t}$. Suppose that $1+t+t^2+\ldots+t^{k-1}=\frac{1-t^k}{1-t}$. We then have

$$\begin{array}{rcl} 1+t+\ldots+t^{k-1}+t^{k+1-1} & = & \frac{1-t^k}{1-t}+t^k \\ & = & \frac{1-t^k+t^k-t^{k+1}}{1-t} \\ & = & \frac{1-t^{k+1}}{1-t}. \end{array}$$

(c) For $n=1$ we have $\frac{1}{1\cdot 2}=\frac{1}{2}$. Suppose that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$. For $n=k+1$, we get

$$\begin{array}{rcl} \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} & = & \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \\ & = & \frac{k(k+2)+1}{(k+1)(k+2)} \\ & = & \frac{k^2+2k+1}{(k+1)(k+2)} \\ & = & \frac{(k+1)^2}{(k+1)(k+2)} \\ & = & \frac{k+1}{k+2}. \end{array}$$

(d) For $n=4$, we get $16-2\cdot 4=8\geq 7=4+3$. Suppose that $k^2-2k\geq k+3$. Then

$$\begin{array}{rcl} (k+1)^2-2(k+1) & = & k^2+2k+1-2k-2 \\ & = & k^2-1 \\ & \geq & 3k+3-1 \\ & = & k+4+2k-2 \\ & = & k+4+2(k-1) \\ & > & (k+1)+3. \end{array}$$

(e) For $n=1$, $7-2=5$ is divisible by 5. Suppose that 5 divides 7^k-2^k. Then

$$\begin{array}{rcl} 7^{k+1}-2^{k+1} & = & 7\cdot 7^k-2\cdot 2^k \\ & = & 2\cdot 7^k-2\cdot 2^k+5\cdot 7^k \\ & = & 2(7^k-2^k)+5\cdot 7^k, \end{array}$$

whence, since the first summand is divisible by 5 by the inductive hypothesis and the second summand is obviously divisible by 5, it follows that the sum is also divisible by 5. ■

Problem 2 *An ice cream parlor serves a sundae for which you can choose one of 20 different flavors, with one of seven different toppings. You can then choose to have whipped cream or not. How many sundaes are possible?*

Solution: $20 \cdot 7 \cdot 2$ possible sundaes. ■

Problem 3 *A witness to a hit-and-run accident tried to memorize the license plate of the car. She remembers that there were two letters followed by three numbers. The first letter was W and the second letter was either C,D,O or Q. The last two numbers were both 7. How many different plate numbers would the police have to check given this information?*

Solution: $4 \cdot 10 = 40$ different license plates. ■

Problem 4 *A certain large city now has five area codes. A phone number consists of the area code followed by seven digits, the first of which cannot be zero or one. How many phone numbers are possible for that city?*

Solution: $5 \cdot 8 \cdot 10^6$ phone numbers. ■

Problem 5 *How many six-letter strings from $\{a, b, c, \dots, z\}$ contain exactly one of the vowels a, e, i, o, u (other letters may be repeated)?*

Solution: $5 \cdot 6 \cdot 21^5$. ■

Problem 6 *Determine the number of license plate codes consisting of three letters followed by three digits that contain a repeated letter or a repeated digit (or both).*

Solution: The total number of license plates consisting of 3 letters followed by 3 digits is $26^3 \cdot 10^3$. Those that contain neither a repeated letter nor a repeated digit are $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8$. Hence the number of all those that contain a repeated letter or a repeated digit is

$$26^3 \cdot 10^3 - 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8.$$

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Problem 7 *There are eight democrats and four Republicans on a senate committee. They must form a subcommittee of five members for a project. How many possible subcommittees are there containing*
(a) exactly two Democrats? (b) at least one republican?

Solution: (a) $\binom{8}{2}\binom{4}{3}$ (b) $\binom{12}{5} - \binom{8}{5}$. ■

Problem 8 *A jar contains 10 red, 12 white and 13 blue balls, all of which are distinct. In how many ways can seven balls be selected so that there are at least two balls of each color?*

Solution: $\binom{10}{2}\binom{12}{2}\binom{13}{2}(8+10+11)$. ■

Problem 9 *(a) How many ordered five-letter sequences can be made using the letters A, A, B, C and D? An example of an ordered sequence using these letters is $BADAC$.*
(b) Generalize part (a) to the case in which you are given n symbols and k of them are the same while the rest are all distinct. How many ordered n-letter sequences are there?

Solution: (a) $\frac{5!}{2!}$. (b) $\frac{n!}{k!}$ ■

Problem 10 *Prove the following identity:* $1 \times 3 \times 5 \times \ldots \times (2n-1) = \frac{(2n)!}{n!2^n}$.

Solution: For $n = 1$, $1 = \frac{(2 \cdot 1)!}{1!2^1}$. Suppose that $1 \times 3 \times 5 \times \ldots \times (2k-1) = \frac{(2k)!}{k!2^k}$. For $n = k+1$, we get

$$\begin{aligned} 1 \times 3 \times 5 \times \ldots \times (2k-1) \times (2(k+1)-1) &= \frac{(2k)!}{k!2^k} \times (2k+1) \\ &= \frac{(2k)!(2k+1)(2k+2)}{k!2^k(2k+2)} \\ &= \frac{(2k+2)!}{k!2^k 2(k+1)} \\ &= \frac{(2(k+1))!}{(k+1)!2^{k+1}}. \end{aligned}$$

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Problem 11 *Prove by mathematical induction:* $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$.

Solution: For $n = 1$, we get $1 \cdot 1! = 2! - 1$. Suppose that $\sum_{i=1}^{k} i(i!) = (k+1)! - 1$. Then we have

$$\begin{aligned} \sum_{i=1}^{k+1} i(i!) &= (k+1)! - 1 + (k+1)(k+1)! \\ &= (k+1)!(1+k+1) - 1 \\ &= (k+2)! - 1 \\ &= ((k+1)+1)! - 1. \end{aligned}$$

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Problem 12 *Prove by a combinatorial argument that* $\binom{n}{r} = \binom{n-2}{r} + 2\binom{n-2}{r-1} + \binom{n-2}{r-2}$.

Solution: Select r objects out of n objects in two different ways. First select the r out of the pile of all n objects. This can be done in $\binom{n}{r}$ ways. Then take two fixed objects on the side. To select r out of the n objects now we either select all r out of the pile of $n-2$ objects in $\binom{n-2}{r}$ ways or we select $r-2$ objects out of the pile of $n-2$ objects and we also include the two fixed objects in $\binom{n-2}{r-2}$ ways or we select $r-1$ objects out of the pile of the $n-2$ objects and then we select one only out of the 2 fixed objects in $2\binom{n-2}{r-1}$ ways. Thus we have

$$\binom{n}{r} = \binom{n-2}{r} + 2\binom{n-2}{r-1} + \binom{n-2}{r-2}.$$

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Problem 13 *Find appropriate choices for a and b in the binomial theorem in order to evaluate the given quantity as a function of n:*
(a) $\sum_{k=0}^{n} \binom{n}{k} 5^k$ *(b)* $\sum_{k=0}^{n} \binom{n}{k} 10^{n-k}$.

Solution: (a)

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}5^k &= \sum_{k=0}^{n}\binom{n}{k}5^k1^{n-k}\\ &= (5+1)^n\\ &= 6^n.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}10^{n-k} &= \sum_{k=0}^{n}\binom{n}{k}10^{n-k}1^k\\ &= (10+1)^n\\ &= 11^n.\end{aligned}$$

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