HOMEWORK 3 - MATH 351
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Problem 1 *A* **forest** *is a graph whose components are trees. There are six nonisomorphic forests that have four vertices. Find them.*

Solution: Let $V = \{0,1,2,3\}$. The forests are $G_1 = \langle V, E_1\rangle$, with $E_1 = \{\{0,1\},\{0,2\},\{0,3\}\}$, $G_2 = \langle V, E_2\rangle$, with $E_2 = \{\{0,1\},\{0,2\},\{1,3\}\}$, $G_3 = \langle V, E_3\rangle$, with $E_3 = \{\{0,1\},\{0,2\}\}$, $G_4 = \langle V, E_4\rangle$, with $E_4 = \{\{0,1\},\{2,3\}\}$, $G_5 = \langle V, E_5\rangle$, with $E_5 = \{\{0,1\}\}$, and $G_6 = \langle V, E_6\rangle$, with $E_6 = \emptyset$. ■

Problem 2 *There are eleven nonisomorphic trees that have seven vertices. Draw them.*

Solution: Set $V = \{0,1,2,3,4,5,6\}$. Then the eleven trees are $G_1 = \langle V, E_1\rangle, \ldots, G_{11} = \langle V, E_{11}\rangle$, with

$$E_1 = \{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\}\}, E_2 = \{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{4,6\}\},$$

$$E_3 = \{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{3,5\},\{4,6\}\}, E_4 = \{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{3,5\},\{3,6\}\},$$

$$E_5 = \{\{0,1\},\{1,2\},\{2,3\},\{2,4\},\{3,5\},\{3,6\}\}, E_6 = \{\{0,1\},\{1,2\},\{1,3\},\{2,4\},\{4,5\},\{4,6\}\},$$

$$E_7 = \{\{0,1\},\{1,2\},\{2,3\},\{2,4\},\{2,5\},\{3,6\}\}, E_8 = \{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{2,5\},\{5,6\}\},$$

$$E_9 = \{\{0,1\},\{1,2\},\{2,3\},\{2,4\},\{2,5\},\{2,6\}\}, E_{10} = \{\{0,1\},\{1,2\},\{1,3\},\{2,4\},\{2,5\},\{2,6\}\},$$

$$E_{11} = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\}\}.$$

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Problem 3 *Suppose that a tree has 50 vertices. How many edges does it have?*

Solution: It has $50 - 1 = 49$ edges. ■

Problem 4 *Show that if a forest F contains c trees and a total of n vertices, then the number of edges in F is $n - c$.*

Solution: By induction on the number of trees in the forest. If the forest has $c = 1$ tree, then it has $n - 1 = n - c$ edges. Suppose that a forest with $c < k$ trees has $n - c$ edges. Now let F be a forest with k trees and e edges. Create a new forest F' by adding an edge between 2 of forest F's trees. Then F' has n vertices, $e + 1$ edges and $k - 1$ trees. Thus, by the induction hypothesis, $e + 1 = n - (k - 1) = n - k + 1$, i.e., $e = n - k$, as was to be shown. ■

Problem 5 *Prove that if T_1 and T_2 are trees with n_1 and n_2 vertices, respectively, then the join $T_1 + T_2$ has $n_1 + n_2$ vertices and $(n_1 + 1)(n_2 + 1) - 3$ edges.*

Solution: The join inherits $n_1 - 1$ edges from T_1, $n_2 - 1$ edges from T_2 and has $n_1 n_2$ new edges joining a vertex from T_1 with a vertex from T_2. Therefore the join will have $n_1 - 1 + n_2 - 1 + n_1 n_2 = (n_1 + 1)(n_2 + 1) - 3$ edges. ■

Problem 6 *A rooted tree is called* **binary** *if each vertex has at most two children. A finite binary tree is called* **complete** *if each vertex, except each leaf, has exactly two children. How many vertices are there in a complete binary tree of height k? How many leaves are there?*

Solution: We show by induction on the height n that a complete binary tree of height n has $2^{n+1}-1$ vertices and 2^n leaves.

If a complete binary tree has height $n=0$, Then it has $1=2^{0+1}-1$ vertex which happens to be $1=2^0$ leaf.

Suppose the statement is true for $n=k$. We show that it is true for $n=k+1$. Let T be a complete binary tree of height $k+1$. If we delete all its leaves and their incident edges we obtain a complete binary tree T' of height k. Thus, by the induction hypothesis T' has $2^{k+1}-1$ vertices and 2^k leaves. Each leaf of T' has two children in T that happen to be the leafs of T, whence T has $2\cdot 2^k=2^{k+1}$ leaves and a total of $2^{k+1}-1+2^{k+1}=2^{k+2}-1=2^{(k+1)+1}-1$ vertices. ■

Problem 7 *If G has n vertices and $n-1$ edges, must G be a tree? Explain.*

Solution: No! For instance $K_1\cup C_3$ has 4 vertices and 3 edges but it is neither connected nor acyclic, so it is not a tree! ■

Problem 8 *Find all nonisomorphic spanning trees for the following graphs:*
(a) The wheel $W_{1,5}$ *(b) $K_{3,3}$*

Solution: (a) Let $W_{1,5}=\langle V,E\rightarrow$, with

$$V=\{0,1,2,3,4,5\},E=\{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\}.$$

It has 5 nonisomorphic spanning trees. These are $G_1=\langle V,E_1\rangle,,\ldots,G_5=\langle V,E_5\rangle$, with

$$E_1=\{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\}\},E_2=\{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{4,5\}\},$$

$$E_3=\{\{0,1\},\{0,2\},\{0,3\},\{3,4\},\{4,5\}\},E_4=\{\{0,1\},\{0,2\},\{2,3\},\{0,4\},\{4,5\}\},$$

and

$$E_5=\{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{4,5\}\}.$$

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Problem 9 *Produce spanning trees of $M(3,3)$ with 2,3,4,5 and 6 end vertices.*

Solution: Let $M(3,3)=\langle V,E\rangle$, with $V=\{0,1,2,3,4,5,6,7,8\}$ and $E=\{\{0,1\},\{1,2\},\{0,3\},\{1,4\},\{2,5\},\{3,4\},\{4,5\},\{3,6\},\{4,7\},\{5,8\},\{6,7\},\{7,8\}\}$. Then $G_2=\langle V,E_2\rangle,\ldots,G_6=\langle V,E_6\rangle$ have $2,\ldots,6$, respectively, end vertices:

$$E_2=\{\{0,1\},\{1,2\},\{2,5\},\{4,5\},\{0,3\},\{3,6\},\{6,7\},\{7,8\}\},$$

$$E_3=\{\{0,1\},\{1,2\},\{2,5\},\{0,3\},\{3,4\},\{3,6\},\{6,7\},\{7,8\}\},$$

$$E_4=\{\{0,1\},\{1,2\},\{0,3\},\{3,4\},\{4,5\},\{3,6\},\{4,7\},\{7,8\}\},$$

$$E_5=\{\{0,1\},\{2,5\},\{0,3\},\{3,4\},\{4,5\},\{3,6\},\{4,7\},\{5,8\}\},$$

$$E_6=\{\{0,1\},\{1,2\},\{1,4\},\{3,4\},\{4,5\},\{4,7\},\{6,7\},\{7,8\}\}.$$

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Problem 10 *(a) For the weighted graphs below, list the edges of the spanning tree in the order in which they would be selected if Kruskal's algorithm were used. Then draw the resulting minimum spanning tree.*
(b) List the edges of the spanning tree in the order in which they would be selected if Prim's algorithm were used beginning at vertex c *in the graph on the left and* beginning at vertex g *in the graph on the right. Then draw the resulting spanning tree.*

Solution: Kruskal's algorithm begins with the list

$$L : \{a,e\},\{b,g\},\{b,d\},\{b,e\},\{d,g\},\{c,d\},\{d,f\},\{c,f\},\{a,b\},\{d,e\},\{e,f\},\{e,g\}.$$

Then it adds edges in the spanning tree as follows:

$$\{a,e\},\{b,g\},\{b,d\},\{b,e\},\{c,d\},\{d,f\}.$$

Prim's algorithm starting at c gives the edges:

$$\{c,d\},\{b,d\},\{b,g\},\{b,e\},\{a,e\},\{d,f\}.$$

Kruskal's algorithm on the second graph starts with the list

$$L : \{a,f\},\{c,f\},\{a,h\},\{f,h\},\{c,g\},\{a,b\},\{a,g\},\{b,c\},\{b,h\},\{g,h\}.$$

It selects edges in the following order:

$$\{a,f\},\{c,f\},\{a,h\},\{c,g\},\{a,b\}.$$

Prim's algorithm starting at g proceeds as follows:

$$\{c,g\},\{c,f\},\{a,f\},\{f,h\},\{a,b\}.$$

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Problem 11 *Draw the seven bipartite graphs (both connected and disconnected) that have four vertices.*

Solution: We set $V = \{0,1,2,3\}$ and we obtain the seven graphs $G_1 = \langle V, E_1\rangle, \ldots, G_7 = \langle V, E_7\rangle$, with

$$E_1 = \emptyset, E_2 = \{\{0,1\}\}, E_3 = \{\{0,1\},\{0,2\}\}, E_4 = \{\{0,1\},\{0,2\},\{0,3\}\},$$

$$E_5 = \{\{0,2\},\{1,3\}\}, E_6 = \{\{0,2\},\{0,3\},\{1,2\}\}, E_7 = \{\{0,2\},\{0,3\},\{1,2\},\{1,3\}\}.$$

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Problem 12 *Draw all connected bipartite graphs with six vertices.*

Solution: Let $V = \{0, 1, 2, 3, 4, 5\}$. We have a list of 16 graphs which we call $G_1 = \langle V, E_1\rangle, \ldots,$ $G_{16} = \langle V, E_{16}\rangle$, with

$$E_1 = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\}\},$$
$$E_2 = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{1,2\},\{1,3\},\{1,4\},\{1,5\}\},$$
$$E_3 = \{\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{1,3\},\{1,4\},\{1,5\}\},$$
$$E_4 = \{\{0,2=\},\{0,3\},\{1,2\},\{1,3\},\{1,4\},\{1,5\}\},$$
$$E_5 = \{\{0,2\},\{1,2\},\{1,3\},\{1,4\},\{1,5\}\},$$
$$E_6 = \{\{0,2\},\{0,3\},\{0,4\},\{1,3\},\{1,4\},\{1,5\}\},$$
$$E_7 = \{\{0,2\},\{0,3\},\{0,4\},\{1,4\},\{1,5\}\},$$
$$E_8 = \{\{0,3\},\{0,4\},\{0,5\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\}\},$$
$$E_9 = \{\{0,4\},\{0,5\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\}\},$$
$$E_{10} = \{\{0,5\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\}\},$$
$$E_{11} = \{\{0,4\},\{0,5\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\}\},$$
$$E_{12} = \{\{0,3\},\{1,3\},\{1,5\},\{2,3\},\{2,4\},\{2,5\}\},$$
$$E_{13} = \{\{0,4\},\{1,3\},\{1,5\},\{2,3\},\{2,5\}\},$$
$$E_{14} = \{\{0,3\},\{1,3\},\{2,3\},\{2,4\},\{2,5\}\},$$
$$E_{15} = \{\{0,3\},\{1,4\},\{2,3\},\{2,4\},\{2,5\}\},$$
$$E_{16} = \{\{0,3\},\{0,4\},\{1,4\},\{1,5\},\{2,5\}\}.$$

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Problem 13 *Suppose that G and H are graphs, at least one of which has an edge. Show that the join $G + H$ is not bipartite.*

Solution: Suppose without loss of generality that G has an edge $\{v, u\}$ and assume to the contrary that $G+H$ is bipartite. Then, since $\{v, u\} \in E(G)$, v and u must belong to different parts of the bipartition of $G+H$. But, if s is a vertex in H, then, if s is in the part of v, $\{v, s\} \notin E(G+H)$, and, if s is in the part of u, $\{u, s\} \notin E(G + H)$, which contradicts the fact that both $\{v, s\}$ and $\{u, s\}$ are edges in $G + H$. ■

Problem 14 *Prove that if a bipartite graph with parts V_1 and V_2 is regular, then $|V_1| = |V_2|$.*

Solution: Suppose that G is k-regular for some k. Then $E(G)$ has $|V_1| \cdot k = |V_2| \cdot k$ elements. Therefore $|V_1| = |V_2|$. ■

Problem 15 *A graph is* **semiregular bipartite** *if vertices in part V_1 all have degree s and vertices in part V_2 all have degree t. Prove that if G is semiregular bipartite, then the line graph $L(G)$ is regular of degree $s + t - 2$.*

Solution: Consider a vertex in $L(G)$. It corresponds to an edge in G. This edge joins a vertex in V_1 with a vertex in V_2. From the V_1 side, it is adjacent to $s-1$ other edges and from the V_2 side, it is adjacent to $t-1$ other edges. Therefore, its degree in $L(G)$ is $s+t-2$. Since the chosen vertex of $L(G)$ was arbitrary, every vertex of $L(G)$ has degree $s+t-2$, whence $L(G)$ is regular of that degree. ■

Problem 16 *Prove that if G_1 and G_2 are bipartite, then so is the Cartesian product $G_1 \times G_2$.*

Solution: Let V_1, V_2 be the two parts in a bipartition of G_1 and U_1, U_2 the two parts in a bipartition of G_2. It is not difficult to see that $(V_0 \times U_0) \cup (V_1 \times U_1), (V_0 \times U_1) \cup (V_1 \times U_0)$ forms a bipartition of $G_1 \times G_2$. ■

Problem 17 *If G is semiregular bipartite with n vertices of degree s and m vertices of degree t, determine the number of edges in G.*

Solution: G must have $ns = mt$ edges. ■

Problem 18 *Prove that $G \times K_2$ always has a perfect matching for all graphs G.*

Solution: A perfect matching can always be achieved by

$$M = \{\{(v,0),(v,1)\} : v \in V(G)\},$$

where $V(K_2) = \{0,1\}$. ■

Problem 19 *Given a positive integer n, construct a graph of order n such that a maximum matching has exactly one edge.*

Solution: An example could be the graph $G = \langle V, E \rangle$, with $V = \{0,1,2,\ldots,n-1\}$ and $E = \{\{0,1\},\{0,2\},\ldots,\{0,n-1\}\}$. Any matching in this graph has at most one edge, whence every maximum matching has one edge. ■

Problem 20 *Let T be a spanning tree of G. Show that a perfect matching for T is also a perfect matching for G Find an example to show that the converse is not true.*

Solution: Since T contains all the vertices of G, every perfect matching of T will cover all of the vertices of G as well.

Let $G = \langle V, E \rangle$, with $V = \{0,1,2,3\}, E = \{\{0,1\},\{1,2\},\{2,3\},\{0,3\}\}$ and let $T = \langle V, E' \rangle$, with $E' = \{\{0,1\},\{1,2\},\{2,3\}\}$. Then $M = \{\{1,2\},\{0,3\}\}$ is a perfect matching for G but not a perfect matching for T. ■

Problem 21 *Find two maximum matchings for each of the following two graphs.*

Solution: (a) We have

$$M_1 = \{\{a, e\}, \{c, d\}\}, M_2 = \{\{a, b\}, \{d, e\}\}.$$

(b) Similarly,

$$M_1 = \{\{f, k\}, \{g, h\}, \{i, j\}\}, M_2 = \{\{f, j\}, \{g, i\}, \{h, k\}\}.$$

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Problem 22 *(a) Let G be the cycle C_{2n} with vertices labeled $1, 2, 3, \ldots, 2n$. How many different maximum matchings does G have? (b) Let H be the cycle C_{2n+1} with vertices labeled $1, 2, \ldots, 2n+1$. How many different maximum matchings does H have?*

Solution: (a) It is not difficult to see that the only two perfect matchings for C_{2n} are

$$M_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2n-1, 2n\}\}, M_2 = \{\{2, 3\}, \{4, 5\}, \ldots, \{2n, 1\}\}.$$

(b) Every maximum matching covers $2n$ out of the $2n+1$ vertices. When the vertex that is left uncovered is decided, then there exists only one way in which the remaining $2n$ vertices may all be covered. Thus, there exist exactly $2n + 1$ maximum matchings. ■

Problem 23 *Applicant A is qualified for jobs a, b, d, e. Applicant B is qualified for b, c, e. Applicant C is qualified for b, d, e. Applicant D is qualified for a, c and e and applicant E is qualified for a and b.*
(a) Draw the associated bipartite graph.
(b) Find a maximum matching to create maximum employment.

Solution: (a)

(b) $\{A, a\}, \{B, c\}, \{C, d\}, \{D, e\}, \{E, b\}$ is a perfect matching providing full employment. ■

Problem 24 *If possible find a system of distinct representatives for each of the collections of sets:*
(a) $A_1 = \{1, 3, 4, 6\}, A_2 = \{2, 4, 5\}, A_3 = \{1, 2, 6\}, A_4 = \{1, 5, 7\}, A_5 = \{1, 3, 4, 5\}$
(b) $A_1 = \{4, 5, 6\}, A_2 = \{1, 2, 3, 5\}, A_3 = \{2, 4, 6, 8\}, A_4 = \{1, 2, 8\}, A_5 = \{3, 6, 8\}, A_6 = \{1, 4, 6\}$.

Solution: (a)$\{1, 2, 6, 5, 4\}$ is a system of distinct representatives for the given collection.
(b) $\{4, 1, 2, 8, 3, 6\}$ is a system of distinct representatives for this collection. ■