HOMEWORK 4 - MATH 351
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Problem 1 *Given a vertex v of C_n, find $e(v)$. Show that v has either one or two eccentric vertices, depending on the parity of n.*

Solution: If $n = 2k$, for some k, then $e(v) = k$. If $e(v) = 2k+1$, for some k, then $e(v) = k$. In the first case v has only one eccentric vertex whereas in the second it has two eccentric vertices. ■

Problem 2 *Find the radius and diameter of C_n. Do the same for P_n. Show that the center of P_n consists of one or two adjacent vertices, depending on the parity of n.*

Solution: All vertices v in C_n have equal eccentricity $e(v) = \lfloor \frac{n}{2} \rfloor$. Therefore $\mathrm{rad}(C_n) = \mathrm{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$.

In P_n the minimum eccentricity of a vertex is $\lfloor \frac{n}{2} \rfloor$ and the maximum eccentricity is $n-1$, whence $\mathrm{rad}(P_n) = \lfloor \frac{n}{2} \rfloor$ and $\mathrm{diam}(P_n) = n-1$. Also note that when n is odd, then only one vertex (the central one) attains the minimum eccentricity, whereas, if n is even, there are two vertices attaining the minimum eccentricity. Therefore the center of P_n consists of one or two adjacent vertices, depending on whether n is odd or even, respectively. ■

Problem 3 *Let H be a spanning subgraph of a graph G. Given vertices u and v in G, show that their distance from one another in H is at least as big as in G, that is $d_H(u,v) \geq d_G(u,v)$.*

Solution: Let P be the path in H that joins u, v and has length $d_H(u,v)$. Since H is a subgraph of G, P is a path in G that joins u, v and has length $d_H(u,v)$. Since by definition the distance of two vertices is the length of the shortest path joining them and P is already a joining path, we get $d_G(u,v) \leq d_H(u,v)$. ■

Problem 4 *Show that $C(C_n) = V(C_n)$; that is show that the center of an n-cycle consists of all its vertices. A graph with this property is called* **self-centered**. *Find another class of self-centered graphs.*

Solution: Given any vertex v in C_n, we have $e(v) = \lfloor \frac{n}{2} \rfloor$, whence $C(C_n) = V(C_n)$.

The class of complete graphs K_n is also a class of self-centered graphs, since, for any vertex v in K_n, $e(v) = 1$. ■

Problem 5 *For the sequential join $G = K_1 + K_1 + \overline{K_2} + K_1 + K_1$, determine $\mathrm{rad}(G), \mathrm{diam}(G), C(G)$ and $P(G)$. Then show that G contains a pair of vertices that are mutually eccentric but not antipodal of one another.*

Solution: Let $G = \langle V, E \rangle$, with $V = \{a,b,c,d,e,f\}$ and

$$E = \{\{a,b\},\{b,c\},\{b,d\},\{c,e\},\{d,e\},\{e,f\}\}.$$

Then we have $e(a) = 4, e(b) = 3, e(c) = e(d) = 2, e(e) = 3$ and $e(f) = 4$. Therefore

$$\mathrm{rad}(G) = 2, \mathrm{diam}(G) = 4, C(G) = \{c,d\}, P(G) = \{a,f\}.$$

Note that c, d are mutually eccentric since their eccentricities equal to 2 and the distance $d_G(c,d) = 2$ but they are not antipodal since their distance is not equal to the diameter of G. ■

Problem 6 *Prove that the wheel $W_{1,n}$ has a spanning tree with one center vertex also has a spanning tree with two center vertices.*

Solution: Let $W_{1,n} = \langle V, E \rangle$, with $V = \{0, 1, \ldots, n\}$ and

$$E = \{\{0,1\}, \{0,2\}, \ldots, \{0,n\}, \{1,2\}, \{2,3\}, \ldots, \{n,1\}\}.$$

First, for any n, $T = \{V, E'\}$, with

$$E' = \{\{0,1\}, \{0,2\}, \ldots, \{0,n\}\}$$

is a spanning tree for $W_{1,n}$, with only one center vertex. To construct a spanning tree with two center vertices we need to consider even and odd n separately.

If n is odd, then $T_o = \langle V, E_o \rangle$, with

$$E_o = \{\{0,1\}, \{0,2\}, \{2,3\}, \ldots, \{n-1,n\}\}$$

is a spanning tree which is isomorphic to a path of odd length and, therefore, has two center vertices.

If n is even, on the other hand, then $T_e = \langle V, E_e \rangle$, with

$$E_e = \{\{0,1\}, \{1,2\}, \ldots, \{\frac{n}{2}-1, \frac{n}{2}\}, \{0, \frac{n}{2}+1\}, \{\frac{n}{2}+1, \frac{n}{2}+2\}, \ldots, \{n-2, n-1\}, \{0,n\}\}$$

is a spanning tree with both 0 and 1 center vertices. ■

Problem 7 *Let G and H be graphs, at least one of which is not complete. Show that* $\operatorname{diam}(G+H) = 2$. *Why must we stipulate that at least one of G or H is not complete?*

Solution: If $u, v \in V(G)$ or $u, v \in V(H)$, then $d_{G+H}(u,v) \leq 2$. If $u \in V(G)$ and $v \in V(H)$, then $d_{G+H}(u,v) = 1$ and the same holds if $v \in V(G)$ and $u \in V(H)$. Therefore $\operatorname{diam}(G+H) \leq 2$. If at least one of G or H is not complete, say G, then there are $u, v \in V(G)$, such that $d_{G+H}(u,v) = 2$. Therefore, in that case $\operatorname{diam}(G+H) = 2$. However, if both G and H are complete graphs then $d_{G+H}(u,v) = 1$, for all $u, v \in V(G+H)$, whence $\operatorname{diam}(G+H) = 1$. ■

Problem 8 *Find the weight of each vertex in the graph below. Find the centroid.*

Solution: The weights are shown in the figure, where the centroid has also been emphasized.

■

Problem 9 *Find all cut vertices and bridges for the graph below:*

Solution: The cut vertices are the vertices e, g and a. The bridges are the edges $\{a, g\}$ and $\{a, k\}$. ■

Problem 10 *Find three different minimal edge cutsets of size 2 for the graph below:*

Solution: The sets $\{\{v, x\}, \{v, w\}\}, \{\{z, x\}, \{z, w\}\}$ and $\{\{z, s\}, \{z, t\}\}$ are edge cutsets of size 2 for the given graph. ■

Problem 11 *Construct a graph G with $\kappa(G) = 2, \lambda(G) = 3, \delta(G) = 4$.*

Solution: Take two copies of K_5 say $G_1 = \langle V_1, E_1 \rangle$, with $V_1 = \{0, 1, 2, 3, 4\}$ and $G_2 = \langle V_2, E_2 \rangle$, with $V_2 = \{5, 6, 7, 8, 9\}$. Then add the edges $\{0, 5\}, \{0, 6\}$ and $\{1, 5\}$. The resulting graph G has $\kappa(G) = 2, \lambda(G) = 3, \delta(G) = 4$. ■

Problem 12 *Construct a graph G with $\kappa(G) = 3, \lambda(G) = 3, \delta(G) = 5$.*

Solution: Take two copies of K_6, say $G_1 = \langle V_1, E_1 \rangle$, with $V_1 = \{0, 1, 2, 3, 4, 5\}$, and $G_2 = \langle V_2, E_2 \rangle$, with $V_2 = \{6, 7, 8, 9, 10, 11\}$. Then add the edges $\{0, 6\}, \{1, 7\}$ and $\{2, 8\}$. The resulting graph G has $\kappa(G) = 3, \lambda(G) = 3, \delta(G) = 5$. ■

Problem 13 *Determine $\kappa(G)$ and $\lambda(G)$ for each of the following graphs:*
(a) The octahedron $\overline{K_2} + C_4$ (b) The sequential join $K_2 + K_3 + \overline{K_2} + \overline{K_3}$. (c) The cartesian product $P_4 \times C_3$.

Solution: $\kappa(\overline{K_2} + C_4) = \lambda(\overline{K_2} + C_4) = 4$.
$\kappa(K_2 + K_3 + \overline{K_2} + \overline{K_3}) = \lambda(K_2 + K_3 + \overline{K_2} + \overline{K_3}) = 2$.
$\kappa(P_4 \times C_3) = \lambda(P_4 \times C_3) = 3$. ■

Problem 14 *Draw the line graph $L(W_{1,4})$. Then find $\kappa(L(W_{1,4}))$ and $\lambda(L(W_{1,4}))$.*

Solution: $\kappa(L(W_{1,4})) = \lambda(L(W_{1,4})) = 4$. ■

Problem 15 *Prove that every k-connected graph on n vertices has at least $\frac{nk}{2}$ edges.*

Solution: If G is k-connected and has n vertices, then, by $\kappa(G) \leq \delta(G)$, we get $\deg(v) \geq k$, for all $v \in V(G)$, whence $2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq nk$ and, therefore $|E(G)| \geq \frac{nk}{2}$. ■

Problem 16 *Prove that if G is cubic - that is, 3-regular - then $\kappa(G) = \lambda(G)$.*

Solution:[Douglas B. West] Let S be a minimum vertex cut set ($|S| = \kappa(G)$). Since $\kappa(G) \leq \lambda(G)$ holds always, we need only provide an edge cut set of size $|S|$. Let H_1, H_2 be two components of $G - S$. Since S is a minimum vertex cut, each $v \in S$ has a neighbor in H_1 and a neighbor in H_2. Since G is 3-regular, v cannot have two neighbors in H_1 and two in H_2. For each $v \in S$, delete the edge from v to a member of $\{H_1, H_2\}$, where v has only one neighbor. These $\kappa(G)$ edges break all paths from H_1 to H_2 except in the case where a path can enter S via v_1 and leave via v_2. In this case we delete the edge to H_1 for both v_1 and v_2 to break all paths from H_1 to H_2 through $\{v_1, v_2\}$. ■

Problem 17 *Prove that if G is k-connected, then the join $K_1 + G$ is $(k+1)$-connected.*

Solution: Suppose that $K_1 + G$ is not $(k+1)$-connected, i.e., it is at most k-connected. Then there exists a set of k vertices in $K_1 + G$ whose deletion disconnects $K_1 + G$. All these vertices must be vertices coming from G, since, if they contained the vertex coming from K_1, the remaining $k-1$ of those vertices would be a vertex cut set of G contradicting the fact that it is k-connected. But the deletion of any set of vertices from $K_1 + G$ that does not include the vertex coming from K_1 cannot possibly disconnect $K_1 + G$, since the vertex coming from K_1 is connected to every other vertex. ■