FINAL EXAM: SOLUTIONS - MATH 111
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Problem 1 *Find the equation of the line that goes through the point* $(-5, 2)$ *and is perpendicular to the line going through* $(1, 2)$ *and* $(4, 3)$.

Solution:

The line that goes through the points $(1, 2)$ and $4, 3)$ has slope $m = \frac{3-2}{4-1} = \frac{1}{3}$. Hence the line we are seeking has slope -3 and goes through the point $(-5, 2)$. Therefore, its equation is given by the point-slope form $y - 2 = -3(x + 5)$ or $y = -3x - 13$. ■

Problem 2 *Find the domain of the function* $f(x) = \frac{1}{\sqrt{x^2-2x-8}}$.

Solution:

We need to solve the inequality $x^2 - 2x - 8 > 0$. We have $(x - 4)(x + 2) > 0$, whence, the sign table gives $x < -2$ or $x > 4$. Thus $D(f) = \{x : x < -2 \text{ or } x > 4\}$. ■

Problem 3 *A shopping center has a rectangular area of 40,000 square yards enclosed on three sides for a parking lot. The length is 200 yards more than twice the width. Find the length and width of the lot.*

Solution:

Let l, w denote the length and the width, respectively, of the parking lot. Then we have $lw = 40000$ and $l = 2w + 200$. Therefore, we have, by substitution, $(2w + 200)w = 40000$, whence $2w^2+200w-40000 = 0$, i.e., $w^2+100w-20000 = 0$. This gives $(w+200)(w-100) = 0$, and, hence, $w = -200$ or $w = 100$. Since the dimensions cannot be negative, we have $w = 100$ and, as a consequence, $l = 400$. ■

Problem 4 *Find the vertex, the opening direction, the* x*- and* y*-intercepts and sketch the graph of* $f(x) = 4x^2 - 12x - 7$.

Solution:

For the x-coordinate of the vertex, we compute $x = -\frac{b}{2a} = -\frac{-12}{8} = \frac{3}{2}$. the y-coordinate is, therefore $y = f(\frac{3}{2}) = 4\frac{9}{4} - 12\frac{3}{2} - 7 = -16$. Thus the vertex is the point $V = (\frac{3}{2}, -16)$. The parabola opens up since $a = 4 > 0$. For the x-intercepts, we use the quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{12 \pm \sqrt{144 - 4 \cdot 4 \cdot (-7)}}{2 \cdot 4} = \frac{12 \pm 16}{8} = \frac{7}{2}, -\frac{1}{2}.$$

Thus $(-\frac{1}{2}, 0)$ and $(\frac{7}{2}, 0)$ are the x-intercepts. The y-intercept is the point $(0, -7)$. ■

Problem 5 *Solve the equations*

1. $e^{3x-1} = 12$.

2. $2 \ln (y + 1) = \ln (y^2 - 1) + \ln 5$.

Solution:

1. We have $e^{3x-1} = 12$ implies $3x - 1 = \ln 12$, whence $3x = \ln 12 + 1$ or $x = \frac{1}{3}(\ln 12 + 1)$.

2. $2\ln(y+1) = \ln(y^2-1) + \ln 5$ implies that $\ln(y+1)^2 - \ln(y^2-1) = \ln 5$, whence $\ln \frac{(y+1)^2}{y^2-1} = \ln 5$. Therefore $\ln \frac{(y+1)^2}{(y-1)(y+1)} = \ln 5$. Hence $\ln \frac{y+1}{y-1} = \ln 5$. This yields $\frac{y+1}{y-1} = e^5$, whence $y + 1 = e^5(y-1)$, i.e., $y + 1 = e^5 y - e^5$ or $e^5 y - y = e^5 + 1$. Thus $y(e^5 - 1) = e^5 + 1$, which gives the solution $y = \frac{e^5+1}{e^5-1}$.

■

Problem 6 *Solve the following system* **by the Gauss-Jordan method**

$$\left\{ \begin{array}{rcrcrcr} x & + & y & - & z & = & 6 \\ 2x & - & y & + & z & = & -9 \\ x & - & 2y & + & 3z & = & 1 \end{array} \right\}.$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 2 & -1 & 1 & -9 \\ 1 & -2 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & -3 & 3 & -21 \\ 0 & -3 & 4 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 7 \\ 0 & -3 & 4 & -5 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & 16 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 23 \\ 0 & 0 & 1 & 16 \end{array}\right].$$

Thus $(x, y, z) = (-1, 23, 16)$. ■

Problem 7 *On a typical January day in Manhattan the probability of snow is 0.10, the probability of a traffic jam is 0.80 and the probability of snow or of a traffic jam is 0.82. Are these two events independent?*

Solution:
Let S be the event of snowing and J the event of a traffic jam. Then $P(S \cap J) = P(S) + P(J) - P(S \cup J) = 0.10 + 0.80 - 0.82 = 0.08$. Hence we have $P(S)P(J) = 0.10 \cdot 0.80 = 0.08 = P(S \cap J)$. Hence the two events are independent. ■

Problem 8 *During the Iraq war 40% of the population of a certain American city were following the news on CNN, 25% on Fox and the remaining 35% on Public television. Of the CNN viewers 60% were opposed to the war without a second UN resolution, whereas the corresponding percentages for Fox and Public TV were 20% and 75%, respectively. If a viewer in that city is selected at random and is found to support the war without a second UN resolution, what is the probability that he/she followed the news on CNN?*

Solution:

Let C, F, B be the events corresponding to CNN, Fox and Public TV, respectively, while N the event of a viewer being opposed to the war. Then, by Bayes's formula, we have

$$\begin{aligned} P(C|N') &= \frac{P(N'|C)P(C)}{P(N'|C)P(C)+P(N'|F)P(F)+P(N'|B)P(B)} \\ &= \frac{0.4 \cdot 0.4}{0.4 \cdot 0.4+0.8 \cdot 0.25+0.25 \cdot 0.35} \end{aligned}$$

■

Problem 9 *A car dealer has 8 red, 11 gray and 9 blue cars in stock. Ten cars are randomly chosen to be displayed in front of the dealership. Find the probability that*

1. *4 are red and the others are blue.*
2. *at most one is gray and none are blue.*

Solution:

1. $P = \frac{\binom{8}{4}\binom{9}{6}}{\binom{28}{10}}$.
2. $P = 0$.

■

Problem 10 *The probability that a certain machine turns out a defective item is 0.05. What is the probability that in a run of 75 items*

1. *exactly 5 defectives are produced.*
2. *at least 2 defectives are produced.*

Solution:

1. $P(5D) = \binom{75}{5} 0.05^5 0.95^{70}$.
2.

$$\begin{aligned} P(\geq 2D) &= 1 - P(\leq 1D) \\ &= 1 - P(0D) - P(1D) \\ &= 1 - \binom{75}{0} 0.05^0 0.95^{75} - \binom{75}{1} 0.05^1 0.95^{74} \\ &= 1 - 0.05^{75} - 75 \cdot 0.05 \cdot 0.95^{74}. \end{aligned}$$

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