Problem 1  (a) State the Law of Sines.

(b) Let $AD$ be the angle bisector of a triangle $ABC$. Use the law of sines to prove that $\frac{AB}{AC} = \frac{BD}{DC}$.

Solution:

(a) Let $a, b, c$ denote the lengths of the sides $BC, AC$ and $AB$, respectively of the triangle $ABC$ and $R$ the radius of its circumcircle. Then the law of sines states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$  

(b) Denote by $D_1$ the angle $BDA$ and by $D_2$ the angle $CDA$. Then, since $D_1 + D_2 = 180^\circ$, we have $\sin D_1 = \sin D_2$. Now, by the law of sines in the triangle $ABD$ we get $\frac{AB}{\sin D_1} = \frac{BD}{\sin \frac{\pi}{2}}$, i.e., $AB = \frac{BD \sin D_1}{\sin \frac{\pi}{2}}$, and similarly, by the law of sines for the triangle $ADC$ we get $\frac{AC}{\sin D_2} = \frac{DC}{\sin \frac{\pi}{2}}$, i.e., $AC = \frac{DC \sin D_2}{\sin \frac{\pi}{2}}$. Combining these we get:

$$\frac{AB}{AC} = \frac{BD}{DC} \cdot \frac{\sin D_1}{\sin D_2} = \frac{BD}{DC}.$$  

Problem 2  (a) State Ceva’s Theorem.

(b) Show that the altitudes of an acute triangle are concurrent.

Solution:

(a) Ceva’s Theorem says that if $AX, BY$ and $CZ$ are three concurrent Cevians of a triangle $ABC$, then $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

(b) Let $AX, BY$ and $CZ$ be the altitudes of an acute triangle. Then we have $BX = c \cos B, CY = b \cos C, CY = a \cos C, YA = c \cos A, AZ = b \cos A$ and $ZB = a \cos B$. Therefore

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{c \cos B}{b \cos C} \cdot \frac{a \cos C}{c \cos A} \cdot \frac{b \cos A}{a \cos B} = 1.$$  

Therefore, by the converse to Ceva’s Theorem, we get that the three altitudes are concurrent.

Problem 3  (a) Consider the incircle of the triangle $ABC$ touching $BC, AC$ and $AB$ at the points $X, Y$ and $Z$, respectively. Denote by $x$ the length of $AY$ and by $s$ the semiperimeter of $ABC$. Prove that $x = s - a$.  

(b) Show that, if \( r \) is the inradius of \( ABC \), then the area \( (ABC) = sr \).

Solution:

(a) We have

\[
x = \frac{1}{2}(AY + AZ) = \frac{1}{2}(2s - BZ - BX - CX - CY) = \frac{1}{2}(BZ + BX) - \frac{1}{2}(CX + CY) = s - BX - XC = s - (BX + XC) = s - a.
\]

(b) We have

\[
(ABC) = (AIB) + (BIC) + (CIA) = \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br = \frac{1}{2}(a + b + c)r = sr.
\]

Problem 4  
(a) Define the orthic triangle of a triangle \( ABC \).

(b) Show that the altitudes of an acute-angled triangle are the angle bisectors of its orthic triangle.

Solution:

(a) The orthic triangle of a triangle \( ABC \) is the triangle with vertices the feet of the altitudes of the triangle \( ABC \).

(b) Let \( AD, BE \) and \( CF \) be the altitudes of \( ABC \) and \( H \) its orthocenter. we will show that \( FDA = ADE \). The other angle equalities are shown similarly. Since the quadrangle \( FBDH \) has two opposite angles right, it is inscribed in a circle. Thus, \( FDA = FBH \). But \( FBH = FCA \) since these two angles have their sides mutually perpendicular. Now \( FCA = HDE \) because the quadrangle \( HDCE \) is inscribed in a circle by the same argument as before. Combining these three equalities, we obtain the desired \( FDA = ADE \).

Problem 5  
(a) Give the definitions of orthocenter and circumcenter of a triangle.

(b) Given a triangle \( ABC \), draw line \( WV \) through \( A \) parallel to \( BC \), line \( UW \) through \( B \) parallel to \( AC \) and line \( UV \) through \( C \) parallel to \( AB \). Show that the orthocenter of \( ABC \) is the circumcenter of \( UVW \).

Solution:
(a) The orthocenter of the triangle $ABC$ is the common point of intersection of its three altitudes. The circumcenter is the center of the circle that passes through its three vertices, or, equivalently, the point of intersection of the three perpendicular bisectors of its sides.

(b) Notice that $WA = BC = AV$ and similarly $WB = AC = BU$ and $VC = AB = CU$. Therefore, the altitudes of $ABC$ are the perpendicular bisectors of $UVW$, whence the orthocenter of $ABC$ must be the circumcenter of $UVW$. ■