EXAM 1: SOLUTIONS - MATH 325
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Problem 1 *(a) State the Law of Sines.*

(b) Let AD be the angle bisector of a triangle ABC. Use the law of sines to prove that $\frac{AB}{AC} = \frac{BD}{DC}$.

Solution:

(a) Let a, b, c denote the lengths of the sides BC, AC and AB, respectively of the triangle ABC and R the radius of its circumcircle. Then the law of sines states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

(b) Denote by D_1 the angle $B\hat{D}A$ and by D_2 the angle $C\hat{D}A$. Then, since $D_1 + D_2 = 180^o$, we have $\sin D_1 = \sin D_2$. Now, by the law of sines in the triangle ABD we get $\frac{AB}{\sin D_1} = \frac{BD}{\sin \frac{A}{2}}$, i.e., $AB = \frac{BD \sin D_1}{\sin \frac{A}{2}}$, and similarly, by the law of sines for the triangle ADC we get $\frac{AC}{\sin D_2} = \frac{DC}{\sin \frac{A}{2}}$, i.e., $AC = \frac{DC \sin D_2}{\sin \frac{A}{2}}$. Combining these we get:

$$\frac{AB}{AC} = \frac{\frac{BD \sin D_1}{\sin \frac{A}{2}}}{\frac{DC \sin D_2}{\sin \frac{A}{2}}} = \frac{BD}{DC}.$$

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Problem 2 *(a) State Ceva's Theorem.*

(b) Show that the altitudes of an acute triangle are concurrent.

Solution:

(a) Ceva's Theorem says that if AX, BY and CZ are three concurrent Cevians of a triangle ABC, then $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

(b) Let AX, BY and CZ be the altitudes of an acute triangle. Then we have $BX = c\cos B, CX = b\cos C, CY = a\cos C, YA = c\cos A, AZ = b\cos A$ and $ZB = a\cos B$. Therefore

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{c\cos B}{b\cos C} \cdot \frac{a\cos C}{c\cos A} \cdot \frac{b\cos A}{a\cos B} = 1.$$

Therefore, by the converse to Ceva's Theorem, we get that the three altitudes are concurrent. ■

Problem 3 *(a) Consider the incircle of the triangle ABC touching BC, AC and AB at the points X, Y and Z, respectively. Denote by x the length of AY and by s the semiperimeter of ABC. Prove that $x = s - a$.*

(b) *Show that, if r is the inradius of ABC, then the area $(ABC) = sr$.*

Solution:

(a) We have

$$\begin{aligned} x &= AY \\ &= \tfrac{1}{2}(AY + AZ) \\ &= \tfrac{1}{2}(2s - BZ - BX - CX - CY) \\ &= s - \tfrac{1}{2}(BZ + BX) - \tfrac{1}{2}(CX + CY) \\ &= s - BX - XC \\ &= s - (BX + XC) \\ &= s - a. \end{aligned}$$

(b) We have

$$\begin{aligned} (ABC) &= (AIB) + (BIC) + (CIA) \\ &= \tfrac{1}{2}cr + \tfrac{1}{2}ar + \tfrac{1}{2}br \\ &= \tfrac{1}{2}(a + b + c)r \\ &= sr \end{aligned}$$

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Problem 4 *(a) Define the orthic triangle of a triangle ABC.*

(b) *Show that the altitudes of an acute-angled triangle are the angle bisectors of its orthic triangle.*

Solution:

(a) The orthic triangle of a triangle ABC is the triangle with vertices the feet of the altitudes of the triangle ABC.

(b) Let AD, BE and CF be the altitudes of ABC and H its orthocenter. we will show that $F\hat{D}A = A\hat{D}E$. The other angle equalities are shown similarly. Since the quadrangle $FBDH$ has two opposite angles right, it is inscribed in a circle. Thus, $F\hat{D}A = F\hat{B}H$. But $F\hat{B}H = F\hat{C}A$ since these two angles have their sides mutually perpendicular. Now $F\hat{C}A = H\hat{D}E$ because the quadrangle $HDCE$ is inscribed in a circle by the same argument as before. Combining these three equalities, we obtain the desired $F\hat{D}A = A\hat{D}E$. ■

Problem 5 *(a) Give the definitions of orthocenter and circumcenter of a triangle.*

(b) *Given a triangle ABC, draw line WV through A parallel to BC, line UW through B parallel to AC and line UV through C parallel to AB. Show that the orthocenter of ABC is the circumcenter of UVW.*

Solution:

(a) The orthocenter of the triangle ABC is the common point of intersection of its three altitudes. The circumcenter is the center of the circle that passes through its three vertices, or, equivalently, the point of intersection of the three perpendicular bisectors of its sides.

(b) Notice that $WA = BC = AV$ and similarly $WB = AC = BU$ and $VC = AB = CU$. Therefore, the altitudes of ABC are the perpendicular bisectors of UVW, whence the orthocenter of ABC must be the circumcenter of UVW. ■