EXAM 2: SOLUTIONS - MATH 325
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Problem 1 *(a) List the nine points that give the nine-point circle its name.*

(b) Let H be the orthocenter and O the circumcenter of a triangle ABC. Show that $\widehat{HAO} = |\hat{B} - \hat{C}|$.

Solution:

(a) The nine points that give the nine-point circle its name are the three feet of the altitudes, the three medians and the three midpoints of the segments joining the orthocenter to the three vertices of the triangle.

(b) Let B', C' be the midpoints of AC and AB, respectively, and D be the foot of the altitude from A. We have

$$\begin{aligned}\widehat{HAO} &= \widehat{HAC} - \widehat{OAC} \\ &= (90^o - \hat{C}) - \widehat{OC'B'} \\ &= (90^o - \hat{C}) - \widehat{BAD} \\ &= (90^o - \hat{C}) - (90 - \hat{B}) \\ &= \hat{B} - \hat{C},\end{aligned}$$

where the second equality holds since $AB'OC'$ is inscribable in a circle, the third since the two angles have pairwise perpendicular sides and the fourth since the two angles are in a right triangle. ■

Problem 2 *(a) Give the definition of a pedal triangle.*

(b) Let ABC be an equilateral triangle. Let M be a point on its circumcircle lying between B and C. Prove that $MB + MC = MA$.

Solution:

(a) Let P be a point on the interior of a triangle ABC and bring from P the perpendiculars PA', PB' and PC' on the sides BC, AC and AB, respectively, of ABC. Then the triangle $A'B'C'$ is called the *pedal triangle* corresponding to the *pedal point* P.

(b) We have $\widehat{BMD} = \hat{C} = \hat{B} = \widehat{DMC}$. Hence $\frac{MB}{MC} = \frac{DB}{DC}$, i.e., $MB = MC\frac{DB}{DC}$. Also $ADC \approx ACM$, whence $\frac{AC}{AD} = \frac{AM}{AC}$, i.e., $BC\frac{AB}{AD} = AM$. Finally, since $ABD \approx CMD$, we also get $\frac{MC}{DC} = \frac{AB}{AD}$. The three equalities above now yield

$$\begin{aligned}MB + MC &= MC\frac{DB}{DC} + MC \\ &= MC\frac{DB+DC}{DC} \\ &= MC\frac{BC}{DC} \\ &= \frac{MC}{DC}BC \\ &= \frac{AB}{AD}BC = AM.\end{aligned}$$

■

Problem 3 *(a) Define the power of a point with respect to a circle and describe, given a fixed circle, which point has the minimum possible power.*

(b) Let MA and MB be the two tangents from a point M outside a circle to the circle. Let also C, D be the points where a third line through M intersects the same circle. Show that $(AC)(BD) = (AD)(BC)$.

Solution:

(a) Let the circle have center O and radius R and the point be M. Then the power $P(M) = d(M,O)^2 - R^2$. Since $d(M,O)^2 \geq 0$, the power $P(M)$ assumes its minimum possible value when $d(M,O) = 0$, i.e., when $M = O$, in which case $P(O) = -R^2$.

(b) We have $MAC \approx MDA$ and $MBC \approx MDB$ whence

$$\frac{MA}{MD} = \frac{AC}{AD} \quad \text{and} \quad \frac{MB}{MD} = \frac{CB}{BD}.$$

Therefore

$$\frac{AC}{AD} = \frac{MA}{MD} = \frac{MB}{MD} = \frac{CB}{BD},$$

which gives $(AC)(BD) = (AD)(BC)$. ■

Problem 4 *(a) Define the radical axis of two circles.*

(b) Show that if two circles are not intersecting, then their radical axis does not intersect either of them.

Solution:

(a) The radical axis of two circles is the locus of all points on the plane that have equal powers with respect to the two circles.

(b) Suppose for the sake of obtaining a contradiction that the radical axis intersects one of the two circles O at the point M. Since the two circles are not intersecting, this point does not lie on the other circle O'. Thus $P(M) = d(M,O)^2 - R^2 = 0$, whereas $P'(M) = d(M,O')^2 - R'^2 \neq 0$, i.e., $P(M) \neq P'(M)$, and therefore M cannot lie on the radical axis, contrary to hypothesis. ■

Problem 5 *(a) Define the radical center of three circles.*

(b) Prove that the radical axes of three circles whose centers are not collinear, taken two at a time, are concurrent.

Solution:

(a) The radical center of three circles is the common point of intersection of the three radical axes.

(b) Consider the point M of intersection of the radical axes of O and O' and of O' and O''. Then $P(M) = P'(M)$ and $P'(M) = P''(M)$. Therefore $P(M) = P''(M)$ and the point M is also on the radical axis of the circles O and O''. ■