EXAM 3: SOLUTIONS - MATH 325
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Problem 1 *(a) If two lines through a point P meet a circle at points A, B and C, D, respectively, then $PA \cdot PB = PC \cdot PD$.*

(b) If circles are constructed on two cevians as diameters, their radical axis passes through the orthocenter H of the triangle.

Solution:

(a) We have $\widehat{PAC} = 180 - \widehat{BAC} = \widehat{PDB}$ and, similarly, $\widehat{PCA} = \widehat{PBD}$, whence $PAC \approx PDB$ which yields $\frac{PA}{PD} = \frac{PC}{PB}$, i.e., $PA \cdot PB = PC \cdot PD$.

(b) Let ABC be a triangle, BE and CF the two altitudes meeting at the orthocenter H and BX CY two cevians that are the diameters of the two given circles. Note that $BFH \approx CEH$, whence $\frac{BH}{HC} = \frac{HF}{HE}$, i.e., $BH \cdot HE = HC \cdot HF$. But E is ont the circle with diameter BX and F is on the circle with diameter CY. Thus, $BH \cdot HE$ is the power of H with respect to the first circle and $HC \cdot HF$ the power of H with respect to the second circle. This shows that H has equal powers with respect to the two circles and, therefore, is a point of their radical axis. ■

Problem 2 *(a) State Simson's Theorem and then define the* Simson line *of a point with respect to a triangle.*

(b) If the diagonals of a quadrilateral inscribed in a circle are perpendicular, then the distance of the center from one of the sides is equal to one-half the length of the opposite side.

Solution:

(a) Simson's Theorem states that the feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumcircle.

The line containing the feet of the perpendiculars of a point on the circumcircle is called the Simson line of the point with respect to the triangle.

(b) Let $ABCD$ be the cyclic quadrilateral, O the center of the circumcircle and E, F the feet of the perpendiculars from O to the sides BC, AD, respectively. If P is the point of intersection of the two diagonals of the quadrilateral, we have

$$\begin{aligned}
\widehat{BOE} &= \tfrac{1}{2}\widehat{BOC} \\
&= \widehat{BDC} \\
&= 90 - \widehat{DCA} \\
&= 90 - \tfrac{1}{2}\widehat{DOA} \\
&= 90 - \widehat{DOF} \\
&= \widehat{ODF}.
\end{aligned}$$

Hence $DOF \approx OBE$, whence $\frac{OE}{FD} = \frac{OB}{OD} = 1$, i.e., $OE = FD = \frac{1}{2}AD$. ■

Problem 3 *(a) State Ptolemy's Theorem.*

(b) Prove Nagel's Theorem*: The radius of a circumcircle of ABC through A is perpendicular to the line joining the feet of the altitudes through B and C.*

Solution:

(a) If a quadrilateral is inscribed in a circle, the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.

(b) Let BE, CF be the two altitudes and H the orthocenter. Draw the diameter AA' and let G be the point of intersection of AA' with EF. We then have

$$\begin{aligned} \widehat{AFE} + \widehat{BAA'} &= \widehat{ECB} + \widehat{BCA'} \\ &= \widehat{ACA'} \\ &= 90. \end{aligned}$$

Thus $FE \perp AA'$. ■

Problem 4 *(a) Prove the* Ptolemy's Third Theorem*: Let $ABCD$ be a quadrilateral inscribed in a circle with $AB = \alpha, BC = \beta, CD = \gamma, AD = \delta, AC = \mu$ and $BD = \lambda$. Show that $\frac{\lambda}{\mu} = \frac{\alpha\beta+\gamma\delta}{\alpha\delta+\beta\gamma}$.*

(b) Use Ptolemy's Theorem and Ptolemy's Third Theorem to compute the lengths of the diagonals of a quadrilateral inscribed in a circle in terms of the lengths of its four sides. (In other words compute λ and μ above in terms of α, β, γ and δ).

Solution:

(a) Let E be the point of intersection of the two diagonals of a cyclic quadrangle. We have $EAD \approx EBC$, whence $\frac{EA}{ED} = \frac{EB}{EC} = \frac{\alpha}{\gamma}$. Similarly, $EAB \approx EDC$, whence $\frac{EA}{EB} = \frac{ED}{EC} = \frac{\delta}{\beta}$. Therefore, we have

$$\begin{aligned} \frac{\lambda}{\mu} &= \frac{DE+EB}{AE+EC} \\ &= \frac{DE}{AE+EC} + \frac{EB}{AE+EC} \\ &= \frac{1}{\frac{AE}{DE} + \frac{EC}{DE}} + \frac{1}{\frac{AE}{EB} + \frac{EC}{EB}} \\ &= \frac{1}{\frac{\alpha}{\gamma} + \frac{\beta}{\delta}} + \frac{1}{\frac{\delta}{\beta} + \frac{\gamma}{\alpha}} \\ &= \frac{\gamma\delta}{\alpha\delta+\beta\gamma} + \frac{\alpha\beta}{\alpha\delta+\beta\gamma} \\ &= \frac{\alpha\beta+\gamma\delta}{\alpha\delta+\beta\gamma}. \end{aligned}$$

(b) We have $\alpha\gamma + \beta\delta = \lambda\mu$ and $\frac{\alpha\beta+\gamma\delta}{\alpha\delta+\beta\gamma} = \frac{\lambda}{\mu}$. These two give

$$\lambda\mu = \frac{\alpha\beta + \gamma\delta}{\alpha\delta + \beta\gamma}\mu^2 = \alpha\gamma + \beta\delta.$$

Hence

$$\mu = \sqrt{\frac{(\alpha\delta + \beta\gamma)(\alpha\gamma + \beta\delta)}{\alpha\beta + \gamma\delta}}, \quad \lambda = \sqrt{\frac{(\alpha\beta + \gamma\delta)(\alpha\gamma + \beta\delta)}{\alpha\delta + \beta\gamma}}.$$

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Problem 5 *(a) State the Butterfly Theorem.*

(b) State Morley's Theorem.

Solution:

(a) Through the midpoint M of a chord PQ of a circle, any other chords AB and CD are drawn. Chords AD and BC meet PQ at points X and Y. Then M is the midpoint of XY.

(b) The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle. ■