EXAM 4: SOLUTIONS - MATH 325
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Problem 1 *(a) State Pierre Varignon's Theorem and give the definition of Varignon parallelogram.*

(b) Show that, if one diagonal divides a quadrangle into two triangles of equal area, it bisects the other diagonal.

Solution:

(a) The figure formed when the midpoints of the sides of a quadrangle are joined in order is a parallelogram, and its area is half that of the quadrangle. This parallelogram is called the *Varignon parallelogram* of the quadrangle.

(b) Suppose the diagonal BD of the quadrangle $ABCD$ divides the quadrangle into two triangles DAB and BCD of equal area. Since these two triangles have a common base BD, they must have equal altitudes AH and CJ. From the two congruent triangles AHF and CJF, we deduce that $AF = CF$, as was to be shown. ■

Problem 2 *(a) Give Brahmagupta's formula for the area of a cyclic quadrangle and Heron's formula for the area of a triangle.*

(b) If a quadrangle with sides a, b, c, d is inscribed in one circle and circumscribed about another circle, its area K is given by $K^2 = abcd$.

Solution:

(a) If a cyclic quadrangle has sides a, b, c, d and semiperimeter s, its area K is given by $K^2 = (s-a)(s-b)(s-c)(s-d)$. If a triangle has sides a, b, c and semiperimeter s, its area K is given by $K^2 = s(s-a)(s-b)(s-c)$.

(b) Suppose that $AB = a = x+y, BC = b = y+z, CD = c = z+w$ and $DA = d = w+x$. Then $s - a = x+y+z+w-x-y = z+w = c$ and, similarly, $s-b = d, s-c = a$ and $s - d = b$. Therefore $K^2 = (s-a)(s-b)(s-c)(s-d) = abcd$. ■

Problem 3 *(a) Use Heron's formula to find an expression for the length of the altitude of a triangle in terms of the lengths of its sides.*

(b) The sum of the squares of the sides of any quadrangle equals the sum of the squares of the diagonals plus four times the square of the segment joining the midpoints of the diagonals.

Solution:

(a) Let a, b, c be the three sides and s the semiperimeter of a triangle ABC. Then, if h denotes the length of the altitude corresponding to BC, we have

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}ah,$$

whence

$$h = \sqrt{\frac{4s(s-a)(s-b)(s-c)}{a^2}}.$$

(b) Let $ABCD$ be the quadrangle with sides $AB = a, BC = b, CD = c$ and $DA = d$, diagonals $AC = e$ and $BD = f$ and $MN = p$, where M, N are the midpoints of AC, BD, respectively. Then, applying Stewart's Theorem for the median repeatedly, we get

$$p^2 = \frac{1}{4}(2AN^2 + 2CN^2 - e^2) = \frac{1}{4}(2BM^2 + 2DM^2 - f^2)$$

and

$$AN^2 = \frac{1}{4}(2a^2 + 2d^2 - f^2), CN^2 = \frac{1}{4}(2b^2 + 2c^2 - f^2),$$
$$BM^2 = \frac{1}{4}(2a^2 + 2b^2 - e^2), DM^2 = \frac{1}{4}(2c^2 + 2d^2 - e^2).$$

Therefore

$$\begin{aligned} e^2 + f^2 + 4p^2 &= e^2 + f^2 + AN^2 + CN^2 - \tfrac{1}{2}e^2 + BM^2 + DM^2 - \tfrac{1}{2}f^2 \\ &= \tfrac{1}{2}e^2 + \tfrac{1}{2}f^2 + a^2 + b^2 + c^2 + d^2 - \tfrac{1}{2}e^2 - \tfrac{1}{2}f^2 \\ &= a^2 + b^2 + c^2 + d^2. \end{aligned}$$

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Problem 4 *(a) Give the definition of the inner and the outer Napoleon triangles.*

(b) Let ABC be a triangle and O_2 the vertex of the outer Napoleon triangle corresponding to angle B and N_3 the vertex of the inner Napoleon triangle corresponding to angle C. Show that AN_3O_2 is similar to ABC.

Solution:

(a) If equilateral triangles are erected externally on the sides of any triangle, their centers form an equilateral triangle, called the *outer Napoleon triangle* of the original triangle. Similarly, if equilateral triangles are erected internally on the sides of any triangle, their centers form an equilateral triangle that is termed the *inner Napoleon triangle* of the given triangle.

(b) Clearly,

$$\widehat{N_3AO_2} = \widehat{A} - 30^o + 30^o = \widehat{A}.$$

Furthermore, $AN_3 = \frac{\frac{c}{2}}{\cos 30^o} = \frac{c}{\sqrt{3}}$ and, similarly, $AO_2 = \frac{b}{\sqrt{3}}$. Thus $\triangle AN_3O_2 \cong \triangle BAC$. ■

Problem 5 *(a) State Menelaus's Theorem.*

(b) Prove the Theorem of Compagnon*: Divide the hypotenuse $a = BC$ of a right triangle ABC into three equal segments $BD = DE = EC$ and draw the line segments AD, AE. Prove that $AD^2 + AE^2 + DE^2 = \frac{2a^2}{3}$.*

Solution:

(a) If points X, Y, Z on the sides BC, CA, AB (suitably extended) of a triangle ABC are collinear, then

$$\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = 1.$$

Conversely, if this equation holds for points X, Y, Z on the three sides, then these three points are collinear.

(b) Let $DD' \perp AC$ and $EE' \perp AC$. Then $AD' = \frac{AC}{3}, AE' = \frac{2AC}{3}, EE' = \frac{AB}{3}$ and $DD' = \frac{2AB}{3}$. Therefore we get

$$\begin{aligned} AD^2 + AE^2 + DE^2 &= AD'^2 + DD'^2 + AE'^2 + EE'^2 + DE^2 \\ &= \tfrac{b^2}{9} + \tfrac{4c^2}{9} + \tfrac{4b^2}{9} + \tfrac{c^2}{9} + \tfrac{a^2}{9} \\ &= \tfrac{a^2}{9} + \tfrac{4a^2}{9} + \tfrac{a^2}{9} \\ &= \tfrac{2a^2}{3}. \end{aligned}$$

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