HOMEWORK 1: SOLUTIONS - MATH 325
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Problem 1 1. *In any triangle ABC, $(ABC) = \frac{abc}{4R}$.*

2. *Let p and q be the radii of two circles through A, touching BC at B and C, respectively. Then $pq = R^2$.*

Solution:

1. Let h be the length of the altitude from A. Let also A' be the point where the radius OA extended intersects the circumcircle to ABC. Then we have

$$\begin{array}{rcl} (ABC) & = & \frac{1}{2}ah \\ & = & \frac{1}{2}ac\sin B \\ & = & \frac{1}{2}ac\frac{b}{2R} \\ & = & \frac{abc}{4R}. \end{array}$$

2. Let M, N be the centers of the two circles with radii p and q, respectively. Set $\phi = M\hat{B}A$ and $\psi = A\hat{C}N$. We have

$$\begin{array}{rcl} R^2 & = & R \cdot R \\ & = & \frac{b}{2\sin B} \cdot \frac{C}{2\sin C} \\ & = & \frac{b}{2}\frac{1}{\cos\phi}\frac{c}{2}\frac{1}{\cos\psi} \\ & = & \frac{b}{2}\frac{2p}{c}\frac{c}{2}\frac{2q}{b} \\ & = & pq. \end{array}$$

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Problem 2 1. *If X, Y and Z are the midpoints of the sides, the three Cevians are concurrent.*

2. *Cevians perpendicular to the opposite sides are concurrent.*

Solution:

1. We have $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1 \cdot 1 \cdot 1 = 1$. Thus, the midpoints are concurrent.

2. We have $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{c\cos B}{b\cos C} \cdot \frac{a\cos C}{c\cos A} \cdot \frac{b\cos A}{a\cos B} = 1$. Thus, the altitudes are concurrent.

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Problem 3 *Find the ratio of the area of a given triangle to that of a triangle whose sides have the same lengths as the medians of the original triangle.*

Solution:

Suppose that AA', BB' and CC' be the three medians. Let CD be the line segment parallel and equal in length to $A'B'$. We then have $\frac{(ABC)}{(DAA')} = \frac{(CAA')}{(EAA')} = \frac{CA}{EA} = \frac{4}{3}$. Note that $AD = C'C$ and $A'D = BB'$, whence the triangle has sides with the same lengths as the medians of the original triangle. ■

Problem 4 *The square of the length of the angle bisector AL (Figure 1.3D, page 9) is* $bc[1 - (\frac{a}{b+c})^2]$.

Solution:

Lemma 1 (Stewart's Theorem) *Let AX be a Cevian that divides the side BC to the segments BX with length m and XC with length n. Then*

$$b^2m + c^2n = amn + ap^2$$

Proof:

The law of cosines applied to the triangles ABX and AXC gives $c^2 = m^2 + p^2 - 2mp\cos(B\hat{X}A)$ and $b^2 = n^2 + p^2 - 2np\cos(C\hat{X}A)$. These give $\cos(B\hat{X}A) = \frac{m^2+p^2-c^2}{2mp}$ and $\cos(C\hat{X}A) = \frac{n^2+p^2-b^2}{2np}$. Now notice that $B\hat{X}A + C\hat{X}A = 180^o$, whence $\cos(B\hat{X}A) = -\cos(C\hat{X}A)$, i.e., $\cos(B\hat{X}A) + \cos(C\hat{X}A) = 0$. This gives

$$\frac{m^2 + p^2 - c^2}{2mp} + \frac{n^2 + p^2 - b^2}{2np} = 0,$$

$$nm^2 + np^2 - nc^2 + mn^2 + mp^2 - mb^2 = 0,$$

$$b^2m + c^2n = mn(m+n) + p^2(m+n) = amn + ap^2.$$

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Now we continue the **proof of the problem:** Note that, if L is the foot of the angle bisector and E is the point where AL intersects the line that forms an angle equal to B with AC, then the triangles ABL and AEC are similar and $n = LC = CE$. Therefore $\frac{c}{m} = \frac{b}{n}$, whence $m = \frac{ac}{b+c}$ and $n = \frac{ab}{b+c}$. Thus, we have, by Stewart's Theorem,

$$a(p^2 + bc(\frac{a}{b+c})^2) = b^2\frac{ac}{b+c} + c^2\frac{ab}{b+c},$$

whence

$$p^2 + bc(\frac{a}{b+c})^2 = \frac{b^2c}{b+c} + \frac{c^2b}{b+c}$$

and, therefore

$$p^2 = bc - bc(\frac{a}{b+c})^2 = bc[1 - (\frac{a}{b+c})^2].$$

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Problem 5 *The Cevians AX, BY, CZ (Figure 1.4A, page 11) are concurrent (their common point is the* Gergonne point *of ABC).*

Solution:
We have $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{x}{y}\frac{y}{z}\frac{z}{x} = 1$, whence AX, BY and CZ are concurrent. ■

Problem 6 *Prove that $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$.*

Solution:
We have $(ABC) = sr = (s-a)r_a = (s-b)r_b = (s-c)r_c$. Therefore

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{s-a}{(ABC)} + \frac{s-b}{(ABC)} + \frac{s-c}{(ABC)} = \frac{3s-(a+b+c)}{(ABC)} = \frac{s}{(ABC)} = \frac{s}{rs} = \frac{1}{r}.$$

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