HOMEWORK 2: SOLUTIONS - MATH 325
INSTRUCTOR: George Voutsadakis

Problem 1 (O.Bottema†) *Let BM and CN be external bisectors of the angles $B = 12^o$ and $C = 132^o$ of a special triangle ABC, each terminated at the opposite side. Without using trigonometric functions, compare the lengths of the angle bisectors.*

Solution: By calculating angles, we see that $B\hat{N}C = A\hat{B}C = 12^o$, whence NCB is isosceles and $NC = CB$. Also, we have $B\hat{C}M = B\hat{M}C = 48^o$, whence BCM is isosceles as well and $BC = BM$. Thus $NC = BM$. ■

Problem 2 *The orthocenter of an obtuse-angled triangle is an excenter of its orthic triangle.*

Solution:

Let AD, BE and CF be the altitudes and H the orthocenter of ABC. denote by FB' the extension of BF at the other side of F than B. We have $E\hat{F}H = E\hat{A}H = D\hat{A}C = D\hat{F}C = H\hat{F}B'$. thus FH is in fact an external bisector of the orthic triangle. Similarly, EH is an external bisector of the orthic triangle, whence the ortocenter H is an excenter of the orthic triangle. ■

Problem 3 $OH^2 = 9R^2 - a^2 - b^2 - c^2$.

Solution:

Lemma 1 (Length of Median) *Let AM be the median of ABC. Its length is equal to $AM = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$.*

Proof:

We use Stewart's Theorem: We obtain $b^2\frac{a}{2} + c^2\frac{a}{2} = a(AM^2 + \frac{a^2}{4})$, whence $AM^2 + \frac{a^2}{4} = \frac{b^2+c^2}{2}$, i.e., $AM^2 = \frac{2b^2+2c^2-a^2}{4}$. Therefore $AM = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ ■

Now back to the **proof of the problem:** We refer to Figure 1.7A of page 18. Apply Stewart's theorem to the triangle AOA'. we get $(AA')(OG^2 + AG \cdot GA') = OA^2GA' + OA'^2GA$, whence, denoting by n one third of the median AA', $3n(OG^2 + 2n^2) = R^2n + 2n(R^2 - \frac{a^2}{4})$, i.e., $3OG^2 + 6n^2 = R^2 + 2R^2 - \frac{a^2}{2}$. Hence $OG^2 = R^2 - 2n^2 - \frac{a^2}{6}$. This we use as follows:

$$\begin{aligned} OH^2 &= (3OG)^2 \\ &= 9R^2 - 18n^2 - \tfrac{3a^2}{2} \\ &= 9R^2 - \tfrac{3a^2}{2} - 2(3n)^2 \\ &= 9R^2 - \tfrac{3a^2}{2} - \tfrac{2b^2+2c^2-a^2}{2} \\ &= 9R^2 - a^2 - b^2 - c^2. \end{aligned}$$

■

Problem 4 $DA' = \frac{|b^2-c^2|}{2a}$.

Solution:

We have $AB^2 - BD^2 = AA'^2 - DA'^2$, whence $c^2 - (\frac{a}{2} - DA')^2 = (\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2})^2 - DA'^2$, i.e., $c^2 - \frac{a^2}{4} + aDA' - DA'^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} - DA'^2$. Therefore $aDA' = \frac{b^2-c^2}{2}$ whence $DA' = \frac{b^2-c^2}{2a}$. ■

Problem 5 *1. The quadrilateral $AKA'O$ (Figure 1.8A, page 21) is a parallelogram.*

2. In the nine-point circle (Figure 1.8B, page 21), the points K, L, M bisect the respective arcs EF, FD, DE.

Solution:

1. Both AK and OA' are perpendicular to BC and have length equal a third of the length of the altitude AD. therefore $AKA'O$ is a parallelogram.

2. Since N is the center of the nine point circle, the points K and A' are diametrically opposite. Also $A'K$ is parallel to OA and OA is perpendicular to EF. Thus $A'K$ is also perpendicular to the chord EF and, as a consequence bisects the arc EF. ■

Problem 6 *The circumcircle of ABC is the nine-point circle of $I_aI_bI_c$.*

Solution:

ABC is the orthic triangle of $I_aI_bI_c$, whence a, B, c all lie on the 9 point circle of $I_aI_bI_c$ and, since three points uniquely determine a circle, that circle must be the nine point circle of $I_aI_bI_c$. ■