HOMEWORK 3: SOLUTIONS - MATH 325
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Problem 1 *If a Cevian AQ of an equilateral triangle ABC is extended to meet the circumcircle at P, then*

$$\frac{1}{PB}+\frac{1}{PC}=\frac{1}{PQ}.$$

Solution:

Note that $BQP \approx AQC$ and $PQC \approx ABQ$, whence we get the proportionalities

$$\frac{PQ}{PB}=\frac{QC}{AC} \quad \text{and} \quad \frac{PQ}{PC}=\frac{BQ}{AB}.$$

Therefore we have

$$\begin{aligned} \frac{PQ}{PB}+\frac{PQ}{PC} &= \frac{QC}{AC}+\frac{BQ}{AB} \\ &= \frac{QC+BQ}{AB} \\ &= \frac{AB}{AB} \\ &= 1, \end{aligned}$$

whence $\frac{1}{PB}+\frac{1}{PC}=\frac{1}{PQ}$. ■

Problem 2 *If lines PB and PD, outside a parallelogram $ABCD$, make equal angles with the sides BC and DC, respectively, as in Figure 1.9D, page 26, then $C\hat{P}B = D\hat{P}A$.*

Solution:

Let $PQ = ||CB$. Then the two quadrangles $DAQP$ and $CBQP$ are parallelograms. Therefore, $\widehat{BAQ}=\widehat{CDP}=\alpha=\widehat{CBP}=\widehat{BPQ}$. Hence $ABQP$ is inscribed in a circle from which we can deduce that $\widehat{DPA}+\widehat{APC}=\widehat{AQB}=\widehat{APC}+\widehat{CPB}$, the first since the two angles have parallel sides and the second by the inscribed quadrangle. Hence $\widehat{DPA}=\widehat{CPB}$.

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Problem 3 *What is the algebraically smallest possible value that the power of a point can have with respect to a circle of given radius R? Which point has this critical power?*

Solution:

Since the power of a point M is given by $P(M)=d^2-R^2$, where d is the distance of the point M from the center of the circle and R is the radius of the circle, $P(M)$ assumes its minimum value when $d=0$, i.e., when $M=O$, the center of the circle, and this value is $P(O)=-R^2$. ■

Problem 4 *If PT and PU are tangents from P to two concentric circles, with T on the smaller, and if the segment PT meets the larger circle at Q, then $PT^2-PU^2=QT^2$.*

Solution:

Let Q' be the other point of intersection of PQ with the big circle. Then, we have

$$\begin{aligned}
PT^2 - PU^2 &= (PQ - QT)^2 - PQ \cdot PQ' \\
&= PQ^2 + QT^2 - 2PQ \cdot QT - PQ \cdot PQ' \\
&= PQ^2 + QT^2 - PQ(2QT + PQ') \\
&= PQ^2 + QT^2 - PQ(QQ' + PQ') \\
&= PQ^2 + QT^2 - PQ^2 \\
&= QT^2.
\end{aligned}$$

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Problem 5 *When the distance between the centers of two circles is greater than the sum of the radii, the circles have four common tangents. The midpoints of these four line segments are collinear.*

Solution:

The midpoints of all four common tangents have equal powers from both circles. Thus, they all belong to the radical axis of the two circles. ■

Problem 6 *Let $PAB, AQB, ABR, P'BA, BQ'A, BAR'$ be six similar triangles all on the same side of their common side AB. (Three of them are shown in Figure 2.2D, page 34; the rest can be derived by reflection in the perpendicular bisector of the segment AB.) Those vertices of the triangles that do not lie on AB (namely, P, Q, R, P', Q', R') all lie on one circle. (*Hint*: Compare the powers of A and B with respect to the circle PQR.)*

Solution:

The given similarities of the triangles yield the proportionalities

$$\frac{AB}{AQ} = \frac{AR}{AB} \quad \text{and} \quad \frac{AB}{BP} = \frac{BQ}{AB},$$

whence $AR \cdot AQ = AB^2 = BP \cdot BQ$. Therefore A and B have the same power with respect to the circle going through PQR. I.e., A and B are equidistant from the center of this circle. This means that the points P', Q' and R' which are symmetric to P, Q and R, respectively, with respect to the perpendicular bisector of AB all lie on the same circle. ■