HOMEWORK 5: SOLUTIONS - MATH 325
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Problem 1 *Let ABC be an equilateral triangle inscribed in a circle with center O, and let P be any point on the circle. Then the Simson line of P bisects the radius OP.*

Solution:

By Theorem 2.72, the Simson line of P bisects the segment joining P to the orthocenter. Since ABC is equilateral, its orthocenter coincides with its circumcenter. Hence, the Simson line bisects the segment OP. ■

Problem 2 *Let PT and PB be two tangents to a circle, AB the diameter through B, and TH the perpendicular from T to AB. Then AP bisects TH.*

Solution:

Consider the point Q, where AT extended intersects PB. We have $\widehat{POB} = \frac{1}{2}\widehat{TOB} = \widehat{TAB}$, whence $QA \parallel PO$. Now, since O is the midpoint of AB, P has to be the midpoint of QB. But, then, since $TH \parallel QB$, we immediately obtain that AP bisects TH. ■

Problem 3 *Let the incircle (with center I) of ABC touch the side BC at X, and let A' be the midpoint of this side. Then the line $A'I$ (extended) bisects AX.*

Solution:

Let Z' be the antidiametric point of X. Bring the tangent $B'C'$ of the incircle at Z', where B' and C' are points on AB and AC, respectively. Now let X' be the point of intersection of AX with $B'C'$ and Z the point of intersection of AZ with BC. Note that $B'C' \perp Z'X \perp BC$, whence $B'C' \parallel BC$ and, therefore the triangles ABC and $AB'C'$ are similar triangles. Therefore, since they share the angle A, the incircle of $AB'C'$ will touch $B'C'$ at the point X'. Then, letting t, t' denote the lengths of the exterior, interior, respectively, common tangents of the two incircles, we obtain

$$\begin{aligned} t &= B'X' + B'Z' \\ &= B'X' + B'X' + t' \\ &= 2B'X' + t', \end{aligned}$$

whence $B'X' = \frac{t-t'}{2}$. The same reasoning also yields that $Z'C' = \frac{t-t'}{2}$. Therefore $B'X' = Z'C'$, and, since $B'C' \parallel BC$, we must also have $BX = ZC$. Therefore $XA' = A'Z$. This combined with $XI = IZ'$ yields that $AZ \parallel A'I$, which, in turn, results in $A'I$ bisecting AX as was to be shown. ■

Problem 4 *For a triangle with angles $3\alpha, 3\beta, 3\gamma$ and circumradius R, Morley's triangle has sides $8R \sin\alpha \sin\beta \sin\gamma$.*

Solution:

Let ABC be the original triangle with $\hat{A} = 3\alpha, \hat{B} = 3\beta$ and $\hat{C} = 3\gamma$ and XYZ be Morley's triangle of ABC. Then, applying the law of sines to the triangles BZX and XBC, respectively, we obtain

$$\frac{ZX}{\sin\beta} = \frac{BX}{\sin(60+\alpha}, \quad \frac{BX}{\sin\gamma} = \frac{a}{\sin(120+\alpha)} = \frac{2R\sin 3\alpha}{\sin(60-\alpha)}.$$

Therefore, we have

$$\begin{aligned} ZX &= \tfrac{2R\sin 3\alpha \sin\beta \sin\gamma}{\sin(60+\alpha)\sin(60-\alpha)} \\ &= \tfrac{2R\sin\alpha\sin\beta\sin\gamma(3-4\sin^2(\alpha)}{\frac{\cos 2\alpha - \cos 120}{2}} \\ &= \tfrac{4R\sin\alpha\sin\beta\sin\gamma(3-4\sin^2\alpha)}{1-2\sin^2\alpha+\frac{1}{2}} \\ &= \tfrac{4R\sin\alpha\sin\beta\sin\gamma(3-4\sin^2\alpha)}{\frac{1}{2}(3-4\sin^2\alpha)} \\ &= 8R\sin\alpha\sin\beta\sin\gamma. \end{aligned}$$

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Problem 5 *The perimeter of the Varignon parallelogram equals the sum of the diagonals of the original quadrangle.*

Solution:

Let $ABCD$ be a quadrangle and denote by P, Q, R and S the midpoints of the sides AB, BC, CD and DA, respectively. Then, $PQ = SR \parallel= \frac{1}{2}AC$ and $PS = QR \parallel= \frac{1}{2}BD$. Hence

$$\begin{aligned} PQ + QR + RS + SP &= \tfrac{1}{2}AC + \tfrac{1}{2}BD + \tfrac{1}{2}AC + \tfrac{1}{2}BD \\ &= AC + BD. \end{aligned}$$

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Problem 6

1. *For a parallelogram the sum of the squares of the sides equals the sum of the squares of the diagonals.*
2. *If an isosceles trapezoid has equal sides of length a, parallel sides of lengths b and c, and diagonals of length d, then $d^2 = a^2 + bc$.*

Solution:

1. We have

$$\begin{aligned} AC^2 + BD^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos\hat{B} + DC^2 + BC^2 - 2(DC)(BC)\cos\hat{C} \\ &= AB^2 + BC^2 + DC^2 + AD^2 - 2(AB)(BC)\cos\hat{B} + 2(AB)(BC)\cos\hat{B} \\ &= AB^2 + BC^2 + CD^2 + DA^2. \end{aligned}$$

2. Consider the isosceles trapezoid $ABCD$ with $AB = b, DC = c, AD = BC = a$ and $AC = BD = d$. Then, applying the law of cosines to the two triangles ABD and BCD, we obtain

$$\cos \hat{C} = \frac{a^2 + c^2 - d^2}{2ac} \quad \text{and} \quad \cos \hat{C} = \frac{d^2 - a^2 - b^2}{2ab}.$$

Hence, we have

$$ba^2 + bc^2 - bd^2 = cd^2 - ca^2 - cb^2,$$

which yields

$$(b + c)d^2 = (b + c)a^2 + bc(b + c),$$

which, in turn, yields

$$d^2 = a^2 + bc.$$

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