HOMEWORK 6: SOLUTIONS - MATH 325
INSTRUCTOR: George Voutsadakis

Problem 1 *For a triangle ABC express the inradius r in terms of $s, s-a, s-b$ and $s-c$.*

Solution:

We have that $(ABC) = sr = \sqrt{s(s-a)(s-b)(s-c)}$. Therefore $s^2r^2 = s(s-a)(s-b)(s-c)$, whence $r^2 = \frac{(s-a)(s-b)(s-c)}{s}$. Therefore

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

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Problem 2 *If a convex quadrangle with sides a, b, c and d is inscribed in a circle of radius R, its area K is given by*

$$K^2 = \frac{(bc+ad)(ca+bd)(ab+cd)}{16R^2}.$$

Solution:

Let $x = AC$ and $y = BD$. Then we have

$$(ABD) = \frac{ady}{4R}, \quad (BCD) = \frac{bcy}{4R}, \quad (ABC) = \frac{abx}{4R}, \quad (ADC) = \frac{cdx}{4R}.$$

The first two yield $K = \frac{(ad+bc)y}{4R}$ and the last two yield $K = \frac{(ab+cd)x}{4R}$. Therefore $K^2 = \frac{(ad+bc)(ab+cd)xy}{16R^2}$. But, by Ptolemy's theorem, we have $xy = ac + bd$, whence

$$K^2 = \frac{(bc+ad)(ca+bd)(ab+cd)}{16R^2}.$$

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Problem 3 *If any point P in the plane of a rectangle $ABCD$ is joined to the four vertices, we have $PA^2 - PB^2 + PC^2 - PD^2 = 0$.*

Solution:

Suppose that the perpendicular to AB and CD through P intersects AB at X and CD at Y. Then, we have

$$\begin{aligned} PA^2 - PB^2 + PC^2 - PD^2 &= PX^2 + XA^2 - PX^2 - XB^2 + PY^2 + YC^2 - PY^2 - YD^2 \\ &= XA^2 - XB^2 + YC^2 - YD^2 \\ &= 0. \end{aligned}$$

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Problem 4 *The outer and the inner Napoleon triangles have the same center.*

Solution:

Let ABC be the given triangle, O_1, O_2 and O_3 be the vertices of the outer Napoleon triangle and N_1, N_2 and N_3 the vertices of the inner Napoleon triangle. Also let X be the midpoint of the side O_2, O_3 and B' the midpoint of AC.

First note that BO_1N_1 is equilateral because $\widehat{N_1BO_1} = 60$ and $BO_1 = BC = BN_1$. Similarly, $CN_1O_1, CN_2O_2, AN_2O_2, AN_3O_3, BN_3O_3$ are equilateral. Second note that $AN_3O_2 \approx ABC$, since $AN_3 = \frac{AB}{\cos 30} = \frac{AB}{\sqrt{3}}, AO_2 = \frac{AC}{\sqrt{3}}$ and $\widehat{N_3AO_2} = \widehat{N_3AC} + \widehat{CAO_2} = \hat{A} - 30 + 30 = \widehat{A}$. Similarly, $AO_3N_2 \approx O_3BN_1 \approx N_3BO_1 \approx N_2O_1C \approx O_2N_1C \approx ABC$. Now note that $\widehat{O_1BO_3} = 60 + B$ and $\widehat{BO_3N_2} = \widehat{BO_3A} - \widehat{N_2O_3A} = 120 - B$, whence $BO_1N_2O_3$ is a parallelogram and this forces $XB' \parallel O_3N_2 \parallel BO_1$. Now $BO_1 = 2XB'$ whence O_1X and BB' intersect at G such that $O_1G = 2GX$ and $BG = 2GB'$. But O_1X and BB' are medians of $O_1O_2O_3$ and ABC, respectively, whence G is the common centroid of these two triangles. We may show similarly that G is also the centroid of $N_1N_2N_3$. ■

Problem 5 *The external bisectors of the three angles of a scalene triangle meet their respective opposite sides at three collinear points.*

Solution:

Let ABC be a scalene triangle and AA' the external bisector of the angle $\hat{A}$. From A' draw thw line parallel to AB intersecting the extension of AC at D. The two triangles BAA' and DAA' are isosceles and congruent, whence $\frac{AB}{AC} = \frac{DA'}{DC} = \frac{DA}{DC} = \frac{A'B}{A'C}$. Thus, the foot of an external bisector cuts the opposite side at a ratio equal to the ratio of the two adjacent sides. Therefore, if BB' and CC' are the remaining two external bisectors of ABC, we have $\frac{A'B}{A'C}\frac{B'C}{B'A}\frac{C'A}{C'B} = \frac{c}{b}\frac{a}{c}\frac{b}{a} = 1$. Therefore, by Menelaus's theorem, the feet of the external bisectors are collinear. ■

Problem 6 *The internal bisectors of two angles of a scalene triangle, and the external bisector of the third angle, meet their respective opposite sides at three collinear points.*

Solution:

Let ABC be a triangle, AA' the external angle bisector and BB' and CC' the two internal angle bisectors. Then we have $\frac{A'B}{A'C}\frac{B'C}{B'A}\frac{C'A}{C'B} = \frac{c}{b}\frac{a}{c}\frac{b}{a} = 1$. Therefore Menelaus's Theorem applies again to give A', B' and C' collinear. ■