EXAM 1: SOLUTIONS - MATH 341
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Problem 1 *1. Let A, B be sets and $f : A \to B, g : B \to A$ be two functions such that $g \circ f = 1_A$. Show that f is one-to-one and that g is onto.*

2. Find an example of two sets A, B and two functions f, g, such that $g \circ f = 1_A$, f is not onto and g is not one-to-one.

Solution:

1. Suppose that $a_1, a_2 \in A$, such that $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. Thus $(g \circ f)(a_1) = (g \circ f)(a_2)$, i.e., $1_A(a_1) = 1_A(a_2)$, which yields $a_1 = a_2$ and f is one-to-one.

   Now let $a \in A$. We have $a = 1_A(a) = (g \circ f)(a) = g(f(a))$. Thus $a \in g(B)$, which shows that g is onto.

2. Take $A = B = \mathbf{N}$ and define $f(n) = n+1$, for all $n \in \mathbf{N}$, and $g(n) = \begin{cases} 0, & \text{if } n = 0 \\ n-1 & \text{if } n \geq 1 \end{cases}$

   Then it is easy to see that $g \circ f = 1_{\mathbf{N}}$ without f being onto and without g being one-to-one. ■

Problem 2 *In $\mathbb{R} \times \mathbb{R}$ consider the relation $\sim$, such that $(x_1, y_1) \sim (x_2, y_2)$ if and only if $3y_1 - 2x_1 = 3y_2 - 2x_2$. Determine whether $\sim$ is an equivalence relation and, if so, describe its equivalence classes.*

Solution:

We have, for all $(x, y) \in \mathbb{R}^2$, $3y - 2x = 3y - 2x$, whence $(x, y) \sim (x, y)$ and $\sim$ is reflexive. If $(x_1, y_1) \sim (x_2, y_2)$, then $3y_1 - 2x_1 = 3y_2 - 2x_2$, whence $3y_2 - 2x_2 = 3y_1 - 2x_1$, i.e., $(x_2, y_2) \sim (x_1, y_1)$, and $\sim$ is symmetric. Finally, suppose that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. Then $3y_1 - 2x_1 = 3y_2 - 2x_2$ and $3y_2 - 2x_2 = 3y_3 - 2x_3$. Therefore $3y_1 - 2x_1 = 3y_3 - 2x_3$, whence $(x_1, y_1) \sim (x_3, y_3)$ and $\sim$ is transitive. Thus $\sim$ is an equivalence relation. To find out what its equivalence classes look like, let us fix a pair $(a, b) \in \mathbb{R}^2$. We have

$$\begin{aligned} [(a,b)] &= \{(x,y) \in \mathbb{R}^2 : (x,y) \sim (a,b)\} \\ &= \{(x,y) \in \mathbb{R}^2 : 3y - 2x = 3b - 2a\} \\ &= \{(x,y) \in \mathbb{R}^2 : y = \tfrac{2}{3}x + \tfrac{3b-2a}{3}\}. \end{aligned}$$

Thus, each of the equivalence classes is a straight line with slope $\frac{2}{3}$. $\mathbb{R}^2$ is partitioned by $\sim$ into infinitely many parallel straight lines. ■

Problem 3 *Show that n is prime if and only if in $\mathbf{Z}_n$, $[r][s] = [0]$ always implies $[r] = [0]$ or $[s] = [0]$.*

Solution:

First we show that if n is prime, then in $\mathbf{Z}_n$, $[r][s] = [0]$ always implies $[r] = [0]$ or $[s] = [0]$. So, suppose that $[rs] = [0]$. This gives $n\backslash rs$. But, since n is a prime, $n\backslash r$ or $n\backslash s$. Therefore $[r] = [0]$ or $[s] = [0]$.

Conversely, suppose that in $\mathbf{Z}_n$, $[r][s] = [0]$ always implies $[r] = [0]$ or $[s] = [0]$. We need to show that n is a prime. We will show that the contrapositive holds, i.e., that, if n is not prime, then, there exist r, s, such that $[r][s] = [0]$ with $[r] \neq [0]$ and $[s] \neq [0]$. So, suppose that n is not prime. Then there exist $r, s \in \mathbf{Z}_+$, such that $n = rs$ and $1 < r, s < n$. This shows that $[r] \neq [0], [s] \neq [0]$ and $[r][s] = [rs] = [n] = [0]$. ∎

Problem 4 *Find all the complex fourth roots of* $-1 - i$.

Solution:

Suppose that $z = r(\cos\phi + i\sin\phi)$ is a complex fourth root of $-1 - i$. Then we have

$$z^4 = r^4(\cos(4\phi) + i\sin(4\phi)) = -1 - i = \sqrt{2}(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = \sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}).$$

Thus $r^4 = \sqrt{2}$ and $4\phi = \frac{5\pi}{4} + 2k\pi$, whence $r = \sqrt[8]{2}$ and $\phi = \frac{5\pi}{16} + k\frac{\pi}{2}$. The following table gives the different angles and the resulting solutions:

k	ϕ	z
0	$\frac{5\pi}{16}$	$\sqrt[8]{2}(\cos(\frac{5\pi}{16}) + i\sin(\frac{5\pi}{16}))$
1	$\frac{13\pi}{16}$	$\sqrt[8]{2}(\cos(\frac{13\pi}{16}) + i\sin(\frac{13\pi}{16}))$
2	$\frac{21\pi}{16}$	$\sqrt[8]{2}(\cos(\frac{21\pi}{16}) + i\sin(\frac{21\pi}{16}))$
3	$\frac{29\pi}{16}$	$\sqrt[8]{2}(\cos(\frac{29\pi}{16}) + i\sin(\frac{29\pi}{16}))$

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Problem 5 1. *Show that if $A, B \in M(2, \mathbf{C})$ are invertible, then so is AB.*

2. *Find all matrices A with* $\det(A) = 1$ *in* $M(2, \mathbf{Z}_3)$.

Solution:

1. If $A, B \in M(2, \mathbf{C})$ are invertible, we have $|A|, |B| \neq 0$. But then $|AB| = |A||B| \neq 0$ and AB is also invertible.

2. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2, \mathbf{Z}_3)$ is such that $\det(A) = 1$. Then $ad - bc = 1$, i.e., $ad = 1 + bc$. Thus, we have the following three cases:

   (a) $bc = 0$ and $ad = 1$. This gives $b = 0$ or $c = 0$ and at the same time $a = d = 1$ or $a = d = 2$. Therefore we have the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}.$$

(b) $bc = 1$ and $ad = 2$. In this case $b = c = 1$ or $b = c = 2$ and $a = 1$ and $d = 2$ or $a = 2$ and $d = 1$. Thus we have the matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}.$$

(c) $bc = 2$ and $ad = 0$. In this case $b = 1$ and $c = 2$ or $b = 2$ and $c = 1$ and $a = 0$ or $d = 0$. Thus, we obtain the matrices

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}.$$

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