EXAM 3: SOLUTIONS - MATH 341
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Problem 1 1. *Let $H = \{\sigma \in S_4 : \sigma(2) = 2\}$. Show that H is a subgroup of S_4 and find its order.*

2. *Show that if $\sigma \in S_n$ and $|\sigma| = 2$, then σ is a product of disjoint 2-cycles.*

Solution:

1. Let $\sigma, \tau \in H$. Then $\sigma(2) = 2$ and $\tau(2) = 2$. This, in particular, implies that $\tau^{-1}(2) = 2$. Therefore we have $(\sigma\tau^{-1})(2) = \sigma(\tau^{-1}(2)) = \sigma(2) = 2$. Hence $\sigma\tau^{-1} \in H$ and $H \leq S_4$. The permutations of H leave 2 fixed and act arbitrarily on $\{1, 3, 4\}$. Hence $H \cong S_3$, whence $|H| = |S_3| = 6$.

2. Let $\sigma \in S_n$, such that $|\sigma| = 2$. Then σ cannot be the identity since then its order would be 1 contrary to the hypothesis. Suppose that in the unique cycle decomposition of σ, there exists a cycle of length $k \geq 3$. Then, since the order of σ is the least common multiple of the lengths of its cycles, we would have had $|\sigma| \geq 3$, contrary to hypothesis. Thus, all cycles in the cycle decomposition of σ must be 2-cycles.

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Problem 2 1. *Find all the cosets of $\langle 6 \rangle$ in $\mathbf{Z}_{12}$ and all the cosets of $\langle 6 \rangle$ in the subgroup $\langle 2 \rangle$ of $\mathbf{Z}_{12}$.*

2. *Let H be a subgroup of a group G. Show that for any $a \in G$ we have $|Ha| = |H|$.*

Solution:

1. We have $|6| = 2$, whence $|\langle 6 \rangle| = 2$ and $[\mathbf{Z}_{12} : \langle 6 \rangle] = 6$. The six cosets are

$$\{0, 6\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}.$$

We have $\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$ and $\langle 6 \rangle = \{0, 6\}$, whence $[\langle 2 \rangle : \langle 6 \rangle] = 3$. The three cosets are

$$\{0, 6\}, \{2, 8\}, \{4, 10\}.$$

2. We need to define a 1-1 onto mapping $\phi : H \to Ha$. Consider ϕ, such that $\phi(h) = ha$, for all $h \in H$. To show that ϕ is 1-1, let $h_1, h_2 \in H$. Then $\phi(h_1) = \phi(h_2)$ implies $h_1 a = h_2 a$, whence $h_1 a a^{-1} = h_2 a a^{-1}$, i.e., $h_1 = h_2$ and ϕ is 1-1. ϕ is onto by the definition of Ha.

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Problem 3 *Let H be a subgroup of a group G. Show that the map $a \mapsto a^{-1}$ determines a one-one, onto map between the left cosets of H and the right cosets of H.*

Solution:

The map from the left to the right cosets of H in G is determined by $\phi(aH) = Ha^{-1}$, for all $a \in G$. This is a well-defined map since, if $aH = bH$, then $b^{-1}a \in H$, whence $Hb^{-1} = Ha^{-1}$, and, therefore, $\phi(aH) = \phi(bH)$.

To show that ϕ is 1-1, let $a, b \in G$. Then $\phi(aH) = \phi(bH)$ implies $Ha^{-1} = Hb^{-1}$, whence $a^{-1}b \in H$ and, therefore, $aH = bH$.

To show that it is an onto mapping, let Ha be a right coset. Then $\phi(a^{-1}H) = H(a^{-1})^{-1} = Ha$. ■

Problem 4 1. *Let $\phi : G \to G'$ be a homomorphism, $K = \mathrm{Kern}(\phi)$ and $a \in G$. Show that $\{x \in G : \phi(x) = \phi(a)\} = aK$, the left coset of K to which the element a belongs.*

2. *Show that $U(14) \cong U(18)$.*

Solution:

1. First, suppose that $g \in \{x \in G : \phi(x) = \phi(a)\}$. Then $\phi(g) = \phi(a)$, whence $\phi(a^{-1}g) = \phi(a)^{-1}\phi(g) = e'$, i.e., $a^{-1}g \in K$. Therefore $g \in aK$. Suppose, for the reverse inclusion that $g \in aK$. Then $a^{-1}g \in K$, whence $\phi(a^{-1}g) = e'$, i.e., $\phi(a)^{-1}\phi(g) = e'$ and, therefore, $\phi(a) = \phi(g)$.

2. We have $U(14) = \{1, 3, 5, 9, 11, 13\}$ and $U(18) = \{1, 5, 7, 11, 13, 17\}$. Both are cyclic of order 6: $U(14) = \langle 3 \rangle$ and $U(18) = \langle 5 \rangle$. Hence they are isomorphic.

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Problem 5 1. *For $r \in \mathbb{R}^*$ let $rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$. Show that $H = \{rI : r \in \mathbb{R}^*\}$ is a normal subgroup of* $\mathrm{GL}(2, \mathbb{R})$.

2. *Let $Z(G)$ be the center of a group G. Show that if the index $[G : Z(G)] = p$, a prime, then G is Abelian.*

Solution:

1. To show that it is a subgroup, let $rI, sI \in H$. Then $(rI)(sI)^{-1} = (rI)(s^{-1}I) = (rs^{-1})I \in H$.

   To show it is normal, let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R})$, $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \in H$. Then, we have

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} &= \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \\
&= r \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \\
&= rI \\
&\in H
\end{aligned}.$$

2. In Problem 5 of Homework 6, it was shown that if $G/Z(G)$ is cyclic then G is abelian. Now, if $[G : Z(G)] = p$ a prime, then $G/Z(G)$ is a group of prime order and therefore cyclic. Therefore G is abelian.

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