EXAM 4: SOLUTIONS - MATH 341
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Problem 1 1. *Let G be a group, $H \lhd G, \phi \in \mathrm{Aut}(G)$. Show that $\phi(H) \lhd G$.*

2. *Let G_1, G_2 be groups. Show that $Z(G_1 \times G_2) \cong Z(G_1) \times Z(G_2)$.*

Solution:

1. Suppose $x \in \phi(H)$ and $g \in G$. We need to show that $gxg^{-1} \in \phi(H)$. Since $x \in \phi(H)$, there exists $h \in H$, such that $x = \phi(h)$. Furthermore, since ϕ is an automorphism, it is onto. Hence there is $g' \in G$, such that $g = \phi(g')$. Hence $gxg^{-1} = \phi(g')\phi(h)\phi(g')^{-1} = \phi(g'hg'^{-1}) \in \phi(H)$, since $h \in H$ and $H \lhd G$ implies $g'hg'^{-1} \in H$.

2. We show that $Z(G_1 \times G_2) \cong Z(G_1) \times Z(G_2)$. We have $(z_1, z_2) \in Z(G_1) \times Z(G_2)$ if and only if $z_1 \in Z(G_1)$ and $z_2 \in Z(G_2)$ if and only if, for all $g_1 \in G_1$ and $g_2 \in G_2$, $g_1z_1 = z_1g_1$ and $g_2z_2 = z_2g_2$ if and only if, for all $g_1 \in G_1$ and $g_2 \in G_2$, $(g_1z_1, g_2z_2) = (z_1g_1, z_2g_2)$ if and only if for all $(g_1, g_2) \in G_1 \times G_2$, $(g_1, g_2)(z_1, z_2) = (z_1, z_2)(g_1, g_2)$ if and only if $(z_1, z_2) \in Z(G_1 \times G_2)$. So, as sets $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$. That their operations yield isomorphic groups under the identity function is very easy to see.

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Problem 2 1. *Find the largest order of any element in $\mathbf{Z}_{21} \times \mathbf{Z}_{35}$.*

2. *Let G be an Abelian group and $\phi : G \to G$ a homomorphism such that $\phi(\phi(g)) = g$ for all $g \in G$. Show that $G \cong \phi(G) \times \mathrm{Kern}\phi$.*

Solution:

1. We now that the largest order is provided by the order of $(1, 1)$. Thus, we have

$$|(1,1)| = \mathrm{lcm}(|1|, |1|) = \mathrm{lcm}(21, 35) = 105.$$

2. We show that $G = \phi(G) \oplus \mathrm{Ker}(\phi)$. Then, by Theorem 3.3.8, we will have $G \cong \phi(G) \times \mathrm{Ker}(\phi)$. Since G is abelian, we have $\phi(G) \lhd G$ and $\mathrm{Ker}(\phi) \lhd G$. Therefore, it suffices to show that $\phi(G) \cap \mathrm{Ker}(\phi) = \{1\}$ and $\phi(G)\mathrm{Ker}(\phi) = G$.

   For the first equality, let $x \in \phi(G) \cap \mathrm{Ker}(\phi)$. Then, there exists $g \in G$, such that $x = \phi(g)$ and $\phi(x) = 1$. But this gives $\phi(\phi(g)) = 1$, i.e., $g = 1$. Therefore $\phi(G) \cap \mathrm{Ker}(\phi) = \{1\}$.

   For the second equality, notice that $g = \phi(\phi(g)) = \phi(\phi(g))1 \in \phi(G)\mathrm{Ker}(\phi)$. Thus $G \subseteq \phi(G)\mathrm{Ker}(\phi)$ and the reverse inclusion is trivial.

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Problem 3 *1. Let p be a prime. Determine up to isomorphism all Abelian groups of order p^n that contain an element of order p^{n-2}.*

2. Describe the positive integers n such that $\mathbf{Z}_n$ is up to isomorphism the only Abelian group of order n.

Solution:

1. The element of order p^{n-2} has to generate a cyclic group of order p^{n-2}. Thus, one of the direct factors of the group has to be of order at least p^{n-2}. This leaves the following options:

$$\mathbf{Z}_{p^{n-2}} \times \mathbf{Z}_p \times \mathbf{Z}_p, \quad \mathbf{Z}_{p^{n-2}} \times \mathbf{Z}_{p^2}, \quad \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_p, \quad \mathbf{Z}_{p^n}.$$

2. By the fundamental theorem of finite abelian groups, each prime factor must occur with power 1 in the direct decomposition of $\mathbf{Z}_n$. Therefore n must be square-free in order for $\mathbf{Z}_n$ to be the only abelian group of order n up to isomorphism.

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Problem 4 *1. Let R be a ring. If S and T are subrings of R, show that $S \cap T$ is also a subring of R.*

2. Let R be a ring. Show that $(a+b)(a-b) = a^2 - b^2$ for all $a, b \in R$ if and only if R is a commutative ring.

Solution:

1. Let $x, y \in S \cap T$. Then $x, y \in S$ and $x, y \in T$, whence, since both S and T are subrings of R, $x - y \in S, xy \in S$ and $x - y \in T, xy \in T$. Thus $x - y \in S \cap T$ and $xy \in S \cap T$. Hence, by the subring criterion, $S \cap T$ is a subring of R.

2. Suppose that R is commutative. Then $ab = ba$, for all $a, b \in R$. Thus $ab - ba = 0$, whence $a^2 - b^2 = a^2 - ab + ba - b^2 = (a+b)(a-b)$, for all $a, b \in R$. Suppose, conversely, that $(a+b)(a-b) = a^2 - b^2$. Then $a^2 - ab + ba - b^2 = a^2 - b^2$. But then $-ab + ba = 0$, whence $ab = ba$, for all $a, b \in R$, and R is commutative.

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Problem 5 *1. Give an example of a ring R and elements a, b and c in R such that $a \neq 0$, $ab = ac$, but $b \neq c$.*

2. Find all the subdomains of $\mathbf{Z}$.

Solution:

1. Take the ring R to be $\mathbf{Z}_4$ under addition and multiplication. Then $2 \cdot 1 = 2 \cdot 3$, but $1 \neq 3$.

2. All the subdomains of $\mathbf{Z}$ have to contain the multiplicative identity 1. But every subdomain that contains 1 is the entire domain since 1 is a generator for the additive group of the integers. Therefore $\mathbf{Z}$ is the only subdomain of $\mathbf{Z}$.

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