FINAL EXAM - MATH 341

Monday, April 28, 2003

INSTRUCTOR: George Voutsadakis

This is a true/false exam. Read each question very carefully before answering. Each question is worth 1 point.

RULES:

− **Rule** Half a point will be subtracted for each wrong answer.

+ **Rule** If you have at least 60 out of the 90 questions answered correctly, 10 more points will be added to your score for free!!

GOOD LUCK!!

1. (a)
   i. If $f : A \rightarrow B$ and $g : B \rightarrow A$ are such that $g \circ f = 1_A$, then g must be 1-1.
   ii. If $f : A \rightarrow B$ is 1-1 and $g : B \rightarrow C$ is onto, then $g \circ f$ is onto.
   iii. $f : A \rightarrow B$ is said to be invertible if there exists a $g : B \rightarrow A$ such that $g \circ f = 1_A$.
   iv. $f : A \rightarrow B$ is invertible if and only if A and B have the same cardinality.
   v. $|\mathbf{Z}| = |n\mathbf{Z}|$.

   (b)
   i. A relation on a set A is a subset of A.
   ii. A relation $R \subseteq A \times A$ is reflexive if, for all $a, b \in A$, aRb implies bRa.
   iii. If R is an equivalence relation on A, the equivalence class of $a \in A$ is $[a] = \{b \in A : aRb\}$.
   iv. In $\mathbb{R}^2$, the relation $\sim$ defined by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1y_2 = x_2y_1$ is an equivalence relation.
   v. Given a function $f : A \rightarrow B$, the relation $\sim$ on A defined by $a \sim b$ if and only if $f(a) = f(b)$, for all $a, b \in A$, is an equivalence relation.

2. (a)
   i. A set with n elements has 2^{n-1} subsets.
   ii. Given any integers a and $b \geq 1$, there exist unique integers q, r, such that $a = qb + r$.
   iii. Two integers a and b are relatively prime if their only positive common divisor is 1.
   iv. Let a, b, c be integers. If $c \backslash ab$, then $c \backslash a$ or $c \backslash b$.
   v. For any $[r] \in U(n)$, there is an $[s] \in U(n)$, such that $[r][s] = [1]$.

   (b)
   i. The polar representation of $-i$ is $-1(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$.
   ii. Given any complex number $z = r(\cos \theta + i \sin \theta)$, for any positive integer n, we have $z^n = r^n(\cos n\theta + i \sin n\theta)$.

iii. The inverse of $z = 2+3i$, $\frac{1}{z} = \frac{1}{2+3i}$ is not a complex number because it is not written in the form $a + bi$, with $a, b \in \mathbb{R}$.

iv. The complex conjugate of $-\frac{1}{2} + 7i$ is $-2 + \frac{1}{7}i$.

v. The equation $z^5 + 8i = 0$ has 5 complex roots.

3. (a) i. A matrix $A \in M(n, \mathbb{R})$ is invertible iff there exists a matrix $B \in M(n, \mathbb{R})$, such that $A + B = B + A = 0$.

ii. There exist matrices $A, B \in M(2, \mathbb{R})$, such that $AB = I_2$ but $BA \neq I_2$.

iii. For any $A, B \in M(2, \mathbb{R})$, we have $\det(AB) = \det(A)\det(B)$.

iv. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then $\det(A) \neq 0$ and $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} -a & c \\ b & -d \end{bmatrix}$.

v. In $M(2, \mathbf{Z}_7)$, there is no matrix with determinant equal to 2.

(b) i. The n-th complex roots of unity form a group of order $n - 1$ under complex multiplication.

ii. A group $\langle G, * \rangle$ is commutative if and only if $a * (b * c) = (b * c) * a$, for all $a, b, c \in G$.

iii. The dihedral group D_n has order $2n$, if n is even and $2n - 2$ if n is odd.

iv. The rational numbers $\mathbf{Q}$ form a group under multiplication.

v. Let ϕ be the Euler ϕ-function. Then, for all primes p and positive integers k, $\phi(p^k) = p^k(1 - \frac{1}{p})$.

4. (a) i. A nonempty subset H of a group G is a subgroup of G iff $ab^{-1}c \in H$, for all $a, b, c \in H$.

ii. An equation of the form $a * x * b = c, a, b, c \in G$ always has a unique solution in a group $\langle G, * \rangle$.

iii. There is a finite group $\langle G, * \rangle$ with identity e and with an even number of elements, such that $a * a \neq e$, for every $e \neq a \in G$.

iv. There is a non trivial finite cyclic group that has a unique generator.

v. Let G be a group and $a \in G$. The centralizer of a in G consists of all those elements in G that commute with every element in G.

(b) i. Every group has at least one element of every order that divides the order of the group.

ii. There is at least one abelian group of every finite order > 0.

iii. Every group of order ≤ 4 is cyclic.

iv. All generators of $\mathbf{Z}_{20}$ are prime numbers.

v. If G and G' are groups, then $G \cap G'$ is also a group.

5. (a) i. Every cyclic group of order > 2 has at least 2 distinct generators.

ii. There is a subgroup of a cyclic group that is not cyclic.

iii. For every group G and every $a \in G$, a is contained in at least one proper subgroup of G.
iv. The symmetric group S_{10} has 10 elements.
v. S_n is not cyclic for any n.

(b) i. A_3 is a commutative group.
ii. The odd permutations of S_7 form a subgroup of S_7.
iii. Let G be a group and $H \leq G$. Then $|H| \backslash |G|$.
iv. Every group of prime order is abelian.
v. One cannot have left cosets of a finite subgroup of an infinite group.

6. (a) i. A subgroup of a group is a left coset of itself.
ii. For all positive integers a and n, $a^n \equiv a \bmod n$.
iii. For any two groups G and G', there exists a homomorphism of G into G'.
iv. The image of a group of 6 elements under some homomorphism may have 4 elements.
v. A homomorphism of a group G into a group G' is 1-1 if and only if it has an empty kernel.

(b) i. A subgroup H of a group G is normal iff $gHg^{-1} \subseteq H$, for all $g \in H$.
ii. Every subgroup of an abelian group G is a normal subgroup of G.
iii. Every quotient group of a finite group is finite and any quotient group of an abelian group is abelian.
iv. Every quotient group of a non abelian group is non abelian.
v. The subgroup $\langle 6 \rangle + \langle 10 \rangle$ of $\mathbf{Z}_{120}$ has 30 elements.

7. (a) i. Every quotient group of a non cyclic group is non cyclic.
ii. For any group homomorphism $f : G \to G'$ with kernel K, $G/K \cong f(G)$.
iii. The only isomorphism from $\mathbf{Z}_3$ into $\mathbf{Z}_2$ is the trivial one.
iv. $\mathbf{Z}_8$ has 4 automorphisms.
v. There exists an abelian group with at least two inner automorphisms.

(b) i. $\mathbf{Z}_2 \times \mathbf{Z}_4$ is isomorphic to $\mathbf{Z}_8$.
ii. Every abelian group of prime order is cyclic.
iii. Every element of $\mathbf{Z}_4 \times \mathbf{Z}_8$ has order 8.
iv. The order of $\mathbf{Z}_{12} \times \mathbf{Z}_{15}$ is 60.
v. $\mathbf{Z}_3 \times \mathbf{Z}_8$ is isomorphic to S_4.

8. (a) i. Every abelian group of prime power order is cyclic.
ii. Every finite abelian group divisible by 5 has a cyclic subgroup of order 5.
iii. There are 5 different abelian groups of order 360 up to isomorphism.

iv. There are 6 different, up to isomorphism, abelian groups of order p^2q^2, where p, q are primes.

v. $\mathbf{Z}_{180} \times \mathbf{Z}_{42} \times \mathbf{Z}_{35}$ is isomorphic to $\mathbf{Z}_{315} \times \mathbf{Z}_{140} \times \mathbf{Z}_{6}$.

(b) i. Every finite abelian group of order 6 has a cyclic subgroup of order 6.

ii. $\mathbf{Z}_{20} \times \mathbf{Z}_{70} \times \mathbf{Z}_{14}$ is isomorphic to $\mathbf{Z}_{28} \times \mathbf{Z}_{28} \times \mathbf{Z}_{25}$.

iii. A nonempty subset S of a ring R is a subring of R if and only if it is closed under both addition and multiplication.

iv. $S = \{a + b\sqrt{3} : a, b \in \mathbf{Z}\}$ is a subring of $\mathbb{R}$ under the usual operations of addition and multiplication.

v. $S = \{A \in M(2, \mathbb{R}) : \det(A) = 0\}$ is a subring of $M(2, \mathbb{R})$ under addition and multiplication.

9. (a) i. If R is a ring with $a, b, c \in R$ and $a \neq 0$, then $ab = ac$ implies $b = c$.

ii. Let R be a ring. Then $(a + b)^2 = a^2 + 2ab + b^2$, for all $a, b \in R$.

iii. The direct product of two integral domains is an integral domain.

iv. A divisor of zero in a commutative ring with unity can have no multiplicative inverse.

v. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a zero-divisor in $M(2, \mathbf{Z})$.

(b) i. 7 is a zero-divisor in $\mathbf{Z}_{48}$.

ii. The cancellation law holds in the ring $\mathbf{Z}_{15}$.

iii. $\mathbf{Z}[i]$ is an integral domain.

iv. $M(2, \mathbb{R})$ is an integral domain.

v. The equation $a^2 = a$ has exactly two solutions in all commutative rings with identity.