HOMEWORK 1 - MATH 341

DUE DATE: Tuesday, January 28
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work.

GOOD LUCK!!

1. (a) Is the map $f : \mathbf{Q}^* \to \mathbf{Q}^*$, defined by $f(\frac{n}{m}) = \frac{m}{n}$, where $\mathbf{Q}^*$ is the set of nonzero rational numbers, a one to one map?
   (b) Is the map $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 - 4$, an onto map?
   (c) Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, where $f(i) = i + 2$, for $1 \leq i \leq n - 2$, and $f(n-1) = 1$ and $f(n) = 2$ invertible?
   (d) Let $f : A \to B$ and $g : B \to C$ be two maps. Show that
      (i) If $g \circ f$ is onto, then g must be onto.
      (ii) If $g \circ f$ is one-to-one, then f must be one-to-one.
   (e) Show that $|\mathbf{Z} \times \mathbf{Z}| = |2\mathbf{Z} \times 2\mathbf{Z}|$.

2. (a) Determine whether the following relations are equivalent relations and, if so, describe the equivalence classes:
      (i) In $\mathbb{R}$, $a \sim b$ if and only if $|a| = |b|$.
      (ii) In $\mathbb{R}$, $a \sim b$ if and only if $|a - b| \leq 1$.
      (iii) In $\mathbb{R} \times \mathbb{R}$, $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2$.
   (b) Fix an integer n and define on $\mathbf{Z}$ the relation $a \sim b$ if and only if $a - b$ is divisible by n. Show that this is an equivalence relation on $\mathbf{Z}$ and describe the equivalence classes.
   (c) Let $f : S \to T$ be any map and define the relation $\sim$ on S by letting $a \sim b$ if and only if $f(a) = f(b)$. Show that $\sim$ is an equivalence relation on S.

3. (a) The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ is defined by $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$, for $n \geq 1$. Show that $(F_{n+1})^2 - F_n F_{n+2} = (-1)^n$.
   (b) Use the Euclidean algorithm to calculate $\gcd(52, 135)$ and write it as a linear combination of 52 and 135.

(c) Show that if $\gcd(n, r) = 1$, then there exists an integer s such that $\gcd(n, s) = 1$ and $rs \equiv 1 \bmod n$.

(d) Write the multiplication table mod 7 of $U(7)$ and mod 8 of $U(8)$.

4. (a) Calculate the value of i^{38} and express your answer in the form $a + bi, a, b \in \mathbb{R}$.

   (b) Calculate the value of $(1 + i)^7$ and express your answer in the form $a + bi, a, b \in \mathbb{R}$.

   (c) Find all the solutions to the equation $z^4 = -1$.

5. (a) Perform the operation $\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \begin{bmatrix} 2i & i \\ -i & 1 \end{bmatrix}$ in $M(2, \mathbf{C})$.

   (b) Calculate the determinant of $\begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}$ in $\mathbf{Z}_7$.

   (c) Determine whether $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is invertible in $M(2, \mathbf{C})$ and, if so, calculate its inverse.

   (d) Determine whether $\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ is invertible in $M(2, \mathbf{Z}_5)$ and, if so, calculate its inverse.

   (e) Find all the invertible matrices in $M(2, \mathbf{Z}_2)$.