HOMEWORK 10 - MATH 341
DUE DATE: Tuesday, April 22
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work. Correct answers without explanations are worth 0 points.

GOOD LUCK!!

1. (a) Let D be an integral domain and let $S = \{n \cdot 1 : n \in \mathbf{Z}\}$, where 1 is unity in D. Show that S is a subdomain of D and that, if R is any subdomain of D, then $S \subseteq R$.
   (b) Show that $\mathbf{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbf{Z}\}$ is an integral domain.
2. (a) Let R be a ring with at least two elements such that for every nonzero element $a \in R$ there exists a unique element $b \in R$ with $aba = a$. Show that
      i. R has no zero divisors
      ii. $bab = b$
      iii. R has unity.
   (b) Consider the ring $\mathbf{Z}_7$. Show that it is a ring, as in part (a). Then, for any nonzero $a \in \mathbf{Z}_7$, find the corresponding $b \in \mathbf{Z}_7$ with $aba = a$.
3. (a) Let R be a ring with unity $1 \in R$ and S a subring of R with $1 \in S$. Show that if $a \in S$ is a unit in S, then a is a unit in R. Show by example that the converse is not necessarily true.
   (b) Let R_1 and R_2 be commutative rings with unity. Show that the group of units $U(R_1 \times R_2) \cong U(R_1) \times U(R_2)$.
4. (a) An element a of a ring R is called **nilpotent** if, for some $k \geq 1$, we have $a^k = 0$. Show that the set of all nilpotent elements in a commutative ring R form a subring of R.
   (b) Show that if D is an integral domain, then 0 is the only nilpotent element in D.
5. (a) In a ring R an element $a \in R$ is called **idempotent** if $a^2 = a$. Show that in an integral domain, 0 and 1 are the only idempotent elements.

(b) Find all the idempotent elements in $\mathbf{Z}_6, \mathbf{Z}_{12}$ and $\mathbf{Z}_6 \times \mathbf{Z}_{12}$.