HOMEWORK 11 - MATH 341

DUE DATE: Monday, April 28

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Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work. Correct answers without explanations are worth 0 points.

GOOD LUCK!!

1. (a) Prove that $\mathbf{Z}_n \times \mathbf{Z}_m$ and $\mathbf{Z}_{nm}$ are isomorphic rings if and only if n and m are relatively prime.
   (b) For $a, b \in \mathbf{Z}$, let $B(a, b) \in M(2, \mathbf{Z})$ be defined by $B(a, b) = \begin{bmatrix} a & 3b \\ b & a \end{bmatrix}$. Let $S = \{B(a, b) : a, b \in \mathbf{Z}\} \subseteq M(2, \mathbf{Z})$. Show that $S \cong \mathbf{Z}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbf{Z}\}$.
2. (a) Show that $\mathbb{R}$ and $\mathbf{C}$ are not isomorphic rings.
   (b) Show that if $R_1 \cong R_2$ then $\text{char}(R_1) = \text{char}(R_2)$.
3. (a) Find all the maximum ideals in $\mathbf{Z}_{12}$, and in each case describe the quotient ring.
   (b) Let I be an ideal in a ring R. Show that $M(2, I)$ is an ideal in $M(2, R)$.
4. (a) Let R be a ring with unity 1 and I an ideal in R. Show that
      i. If $1 \in I$, then $I = R$.
      ii. If I contains a unit, then $I = R$.
   (b) Let $\phi : F \to R$ be a ring homomorphism where F is a field. Show that ϕ either is one to one or else is the 0-homomorphism.
5. (a) Let R be a commutative ring. The **annihilator** of R is defined as follows: $\text{Ann}(R) = \{a \in R : ax = 0 \text{ for all } x \in R\}$. Show that $\text{Ann}(R)$ is an ideal in R.
   (b) Let R be a commutative ring and I an ideal in R. The **radical** of I is defined as follows: $\text{rad}(I) = \{a \in R : a^n \in I \text{ for all } n \in \mathbf{Z}\}$. Show that $\text{rad}(I)$ is an ideal in R containing I.