HOMEWORK 4: SOLUTIONS - MATH 341
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Problem 1 *(a) Show that for any $n \geq 3$, S_n is a non-Abelian group.*

(b) Show that every permutation $\rho \in S_n$ can be written as a product of 2-cycles of the form $(i \quad i+1)$, where $1 \leq i \leq n$.

Solution:

(a) Consider for instance (12)(123) and (123)(12). We have $(12)(123) = (23) \neq (13) = (123)(12)$.

(b) We know that every permutation $\rho \in S_n$ can be written as a product of 2-cycles. So to show that it can be written as a product of 2-cycles of the form $(i \quad i+1)$, it suffices to show that every 2-cycle may be written as a product of 2-cycles of this special form. So let $(p\ q)$, with $1 \leq p < q \leq n$, be a 2-cycle in S_n. Then it is not difficult to check that

$$(p\ q) = (p\ p{+}1)(p{+}1\ p{+}2)\dots(q{-}2\ q{-}1)(q{-}1\ q)(q{-}2\ q{-}1)\dots(p{+}1\ p{+}2)(p\ p{+}1).$$

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Problem 2 *Consider the regular tetrahedron.*

(a) Find all possible rotations of the regular tetrahedron.

(b) The rotations of the regular tetrahedron correspond to elements of which known group?

Solution:

(a) Let 1234 be the regular tetrahedron. Besides the identity rotation 1, there exist two rotations around each axis through a vertex and the barycenter of the opposite face, one 120^o clockwise and the other 120^o counterclockwise. These 8 rotations fix each one vertex. They are

$$(234), (243), (134), (143), (124), (142), (123), (132).$$

In addition to these, there are three more rotations of 180^o around each of the three axes that pass through the midpoints of two opposite sides. These rotations are

$$(12)(34), (13)(24), (14)(23).$$

(b) The identity and the above 11 rotations are all the permutations of S_4 of even order, since they are even and there are 12 of them. Hence, they correspond to the elements of the alternating group on 4 elements A_4.

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Problem 3 *(a) Find all cosets of the subgroup* $5\mathbf{Z}$ *in* $\mathbf{Z}$.

(b) Find the index of $\langle 10 \rangle$ *in* $\mathbf{Z}_{12}$ *and the index of* $\langle \mu_2 \rangle$ *in* S_3.

Solution:

(a) We have

$$\begin{aligned} [n] &= \{m \in \mathbf{Z} : m - n \in 5\mathbf{Z}\} \\ &= \{m \in \mathbf{Z} : m - n = 5k, \text{ for some } k \in \mathbf{Z}\} \\ &= \{m \in \mathbf{Z} : m = n + 5k, \text{ for some } k \in \mathbf{Z}\} \\ &= \{n + 5k : k \in \mathbf{Z}\}. \end{aligned}$$

Hence, the different cosets correspond to the 5 different remainders of division by 5: $0, 1, 2, 3$ and 4.

(b) We have

$$\mathrm{ind}_{\mathbf{Z}_{12}}(\langle 10 \rangle) = \frac{12}{|10|} = \frac{12}{\frac{12}{\gcd(10,12)}} = \frac{12 \gcd(10,12)}{12} = 2.$$

Similarly,

$$\mathrm{ind}_{S_3}(\langle \mu_2 \rangle) = \frac{|S_3|}{|\mu_2|} = \frac{6}{2} = 3.$$

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Problem 4 *(a) Let* H *be a subgroup of a group* G. *For any* $a, b \in G$, *let* $a \sim b$ *if and only if* $ab^{-1} \in H$. *Show that the relation* $\sim$ *so defined is an equivalence relation on* G, *with equivalence classes the right cosets* Ha *of* H.

(b) Let H *be a subgroup of a group* G. *Show for any* $a \in G$ *that* $aH = H$ *if and only if* $a \in H$.

Solution:

(a) For reflexivity notice that, since $H \leq G$, $e \in H$, whence $aa^{-1} = e \in H$ and $a \sim a$. For symmetry assume that $a \sim b$. Then $ab^{-1} \in H$, whence $(ab^{-1})^{-1} \in H$ and therefore $ba^{-1} \in H$, which implies that $b \sim a$. Finally, for transitivity, if $a \sim b$ and $b \sim c$, then $ab^{-1} \in H$ and $bc^{-1} \in H$, whence, since H is a subgroup, $ab^{-1}bc^{-1} \in H$ and, therefore $ac^{-1} \in H$, i.e., $a \sim c$.

Let $a \in G$. Then

$$\begin{aligned} [a] &= \{b \in G : ba^{-1} \in H\} \\ &= \{b \in G : ba^{-1} = h, \text{ for some } h \in H\} \\ &= \{b \in G : b = ha, \text{ for some } h \in H\} \\ &= \{ha : h \in H\} \\ &= Ha. \end{aligned}$$

(b) Suppose, first, that $aH = H$. Then, since $e \in H$, $ae \in aH = H$ implies that there exists $h \in H$, such that $ae = h$. But $a = ae = h \in H$, whence $a \in H$.

Suppose, conversely, that $a \in H$. We show that $aH \subseteq H$ and that $H \subseteq aH$. For the first inclusion, if $h \in H$, then $ah \in H$, since $a \in H$ and $H \leq G$. Hence $aH \subseteq H$. For the reverse inclusion, if $h \in H$, then $a^{-1}h \in H$, whence $h = a(a^{-1}h) \in aH$ and, therefore, $H \subseteq aH$.

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Problem 5 *(a) Let G be a group with $|G| = p^2$, where p is a prime. Show that every proper subgroup of G is cyclic.*

(b) Let G be a group with $|G| = pq$, where p and q are primes. Show that every proper subgroup of G is cyclic.

Solution:

(a) By Lagrange's Theorem, $H \leq G$ implies that $|H| = 1$ or $|H| = p$ or $|H| = p^2$. In the first case $H = \{e\}$ and therefore H is cyclic. In the second case, every nonidentity element of H generates H, so H is cyclic. In the last case $H = G$ and therefore H is not proper.

(b) As before, by Lagrange's Theorem, $|H| = 1$ or $|H| = p$ or $|H| = q$ or $|H| = pq$. In the first case $H = \{e\}$ which is cyclic. In the second and in the third case, every nonidentity element of H generates H and therefore H is cyclic. In the last case $H = G$ and therefore H is not proper.

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