HOMEWORK 5 - MATH 341
DUE DATE: Tuesday, March 11
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work. Correct answers without explanations are worth 0 points.

GOOD LUCK!!

1. (a) Show that $n^{19} - n$ is divisible by 21 for any integer n.
   (b) Find the remainder of 9^{1573} when divided by 11.

2. (a) Let H be a subgroup of a finite group G and K a subgroup of H. Suppose that the index $[G : H] = n$ and the index $[H : K] = m$. Show that the index $[G : K] = nm$. (Hint: Let $x_i H$ be the distinct cosets of H in G and $y_j K$ the distinct left cosets of K in H. Show that $x_i y_j K$ are the distinct cosets of K in H.)
   (b) Let H and K be subgroups of a group G and for all $a, b \in G$ let $a \sim b$ if and only if $a = hbk$ for some $h \in H$ and $k \in K$. Show that the relation $\sim$ so defined is an equivalence relation. Describe the equivalence classes (which are called **double cosets**).

3. Determine whether the following ϕ is a homomorphism and, in cases where it is, determine its kernel:
   (a) $\phi : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}^*$, where $\mathrm{GL}(2, \mathbb{R})$ is the general linear group of 2×2 invertible matrices and $\phi(A) = \det(A)$.
   (b) $\phi : \mathbf{Z}_7 \to \mathbf{Z}_7$, where $\phi(x) =$ the remainder of x mod 2.

4. (a) Find all possible homomorphisms from $\mathbf{Z}$ to $\mathbf{Z}$.
   (b) Find all possible homomorphisms from $\mathbf{Z}$ onto $\mathbf{Z}$.

5. (a) Show that the dihedral group D_4 contains a subgroup isomorphic to the Klein 4-group V.
   (b) Let $G = \mathrm{GL}(2, \mathbf{Z}_2)$, the general linear group of 2×2 invertible matrices with coefficients in $\mathbf{Z}_2$. Show that $G \cong S_3$.