HOMEWORK 6: SOLUTIONS - MATH 341
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Problem 1 *(a) Find all the normal subgroups in* $\mathrm{GL}(2, \mathbf{Z}_2)$, *the general linear group of* 2×2 *matrices with entries from* $\mathbf{Z}_2$.

(b) Find all the normal subgroups in D_4.

Solution:

(a) In Problem 5 of Homework 5, we saw that $\mathrm{GL}(2, \mathbf{Z}_2) \cong S_3$. So the normal subgroups of $\mathrm{GL}(2, \mathbf{Z}_2)$ are in one to one correspondence with the normal subgroups of S_3. These, as we know, are $\{e\}, A_3$ and S_3. Thus, the normal subgroups of $\mathrm{GL}(2, \mathbf{Z}_2)$ are $\{I\}, \{I, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\}$ and $\mathrm{GL}(2, \mathbf{Z}_2)$.

(b) Recall that $D_4 = \{\rho_0, \rho, \rho^2, \rho^3, \tau, \rho\tau, \rho^2\tau, \rho^3\tau\}$. The following are normal subgroups of D_4: $\{\rho_0\}$, since ρ_0 commutes with all elements of D_4, $Z(D_4) = \{\rho_0, \rho^2\}$ because the elements in the center commute with all other elements of the group, $\langle\rho\rangle = \{\rho_0, \rho, \rho^2, \rho^3\}$, $\langle\rho^2, \tau\rangle = \{\rho_0, \rho^2, \tau, \rho^2\tau\}$ and $\langle\rho^2, \rho\tau\rangle = \{\rho_0, \rho^2, \rho\tau, \rho^3\tau\}$, since all three have order 4 and, therefore, have index 2 in D_4, and, finally, D_4 itself. One checks that no other subgroup of D_4 is normal in D_4.

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Problem 2 *(a) Let* $\phi : G \to G'$ *be a homomorphism and* $H' \lhd G'$. *Show that* $H = \phi^{-1}(H') \lhd G$.

(b) Show that if $H \lhd G$ *and* $K \lhd G$, *then* $H \cap K \lhd G$.

Solution:

(a) Since $H' \leq G'$, we conclude that $\phi^{-1}(H') \leq G$. To show that $\phi^{-1}(H') \lhd G$, we use the normal subgroup test. To this end, let $h \in \phi^{-1}(H')$ and $g \in G$. Then $\phi(h) \in H'$, whence, since $H' \lhd G'$, we get $\phi(g)\phi(h)\phi(g)^{-1} \in H'$, i.e., $\phi(ghg^{-1}) \in H'$, which yields that $ghg^{-1} \in \phi^{-1}(H')$ and $\phi^{-1}(H') \lhd G$, as was to be shown.

(b) Now suppose that $H \lhd G$ and $K \lhd G$ and consider $m \in H \cap K, g \in G$. We apply the normal subgroup test. We have that $gmg^{-1} \in H$, since $m \in H$ and $H \lhd G$, and $gmg^{-1} \in K$, since $m \in K$ and $K \lhd G$. Therefore $gmg^{-1} \in H \cap K$, which yields that $H \cap K \lhd G$.

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Problem 3 *(a) Show that if* $H \lhd G$ *and* $K \lhd G$, *then* $HK \lhd G$.

(b) Let H and K be subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$.

Solution:

(a) We have for $g \in G$, $gHK = HgK = HKg$, where the first equality follows from the normality of H and the second equality follows from the normality of K. Therefore HK is normal in G.

(b) (From *Topics in Algebra* by I.N. Herstein) Suppose, first, that $HK = KH$; that is, if $h \in H$ and $k \in K$, then $hk = k_1h_1$, for some $k_1 \in K, h_1 \in H$. To prove that HK is a subgroup we must verify that it is closed and every element in HK has its inverse in HK. Let's show the closure first; so suppose $x = hk \in HK$ and $y = h'k' \in HK$. Then $xy = hkh'k'$, but since $kh' \in KH = HK, kh' = h_2k_2$ with $h_2 \in H, k_2 \in K$. Hence $xy = h(h_2k_2)k' = (hh_2)(k_2k') \in HK$, and HK is closed. Also $x^{-1} = (hk)^{-1} = k^{-1}h^{-1} \in KH = HK$, so $x^{-1} \in HK$. Thus $HK \leq G$.

On the other hand, if HK is a subgroup of G, then for any $h \in H, k \in K, h^{-1}k^{-1} \in HK$ and so $kh = (h^{-1}k^{-1})^{-1} \in HK$. Thus $KH \subseteq HK$. Now if x is any element of $HK, x^{-1} = hk \in HK$ and so $x = (x^{-1})^{-1} = (hk)^{-1} = k^{-1}h^{-1} \in KH$, so $HK \subseteq KH$. Thus $HK = KH$.

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Problem 4 *Find the order of the indicated element in the indicated quotient group:*

(a) $2 + \langle 6 \rangle$ *in* $\mathbf{Z}_{15}/\langle 6 \rangle$.

(b) $i\langle j \rangle$ *in* $Q_8/\langle j \rangle$.

Solution:

(a) We have that $\langle 6 \rangle = \{0, 3, 6, 9, 12\}$, whence $[\mathbf{Z}_{15} : \langle 6 \rangle] = \frac{15}{5} = 3$. The three cosets are $\langle 6 \rangle, 1 + \langle 6 \rangle$ and $2 + \langle 6 \rangle$ and form a quotient group isomorphic to $\mathbf{Z}_3$. The coset $2 + \langle 6 \rangle$ has order 3 in $\mathbf{Z}_{15}/\langle 6 \rangle$.

(b) Similarly, we have $\langle j \rangle = \{1, -1, j, -j\}$, whence $[Q_8 : \langle j \rangle] = \frac{8}{4} = 2$. The two cosets are $\langle j \rangle$ and $i\langle j \rangle = \{i, -i, k, -k\}$. Thus the group $Q_8/\langle j \rangle$ is isomorphic to $\mathbf{Z}_2$ and the element $i\langle j \rangle$ has order 2.

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Problem 5 *(a) Let $\phi : G \to G'$ be an onto homomorphism with* $\mathrm{Kern}(\phi) = K$, *and let H' be a subgroup of G'. Show that there exists a subgroup H of G such that $K \subseteq H$ and $H/K \cong H'$.*

(b) *Let $Z(G)$ be the center of a group G. Show that $Z(G) \lhd G$ and that, if $G/Z(G)$ is cyclic, then G is Abelian.*

Solution:

(a) Let $H = \phi^{-1}(H')$. We know that since $H' \leq G'$, $H \leq G$. Suppose $k \in K$. Then $\phi(k) = e \in H'$, whence $k \in \phi^{-1}(H')$ and $K \subseteq H$. Consider now the restriction $\phi_H : H \to H'$ of ϕ on H. Since $H = \phi^{-1}(H')$, we get that ϕ_H is onto and $\text{Kern}(\phi_H) = K$, whence, by the First Isomorphism Theorem, $H/K \cong H'$.

(b) To show normality, let $z \in Z(G)$ and $g \in G$. Then $gzg^{-1} = zgg^{-1} = z \in Z(G)$, whence $Z(G) \lhd G$.

Finally, suppose that $G/Z(G)$ is cyclic and let $aZ(G)$ be a generator. Consider now $g, h \in G$. Since the coset space of $Z(G)$ partitions G, there exist $x, y \in Z(G)$, such that $g = a^k x$ and $h = a^l y$, for some integers k, l. Therefore

$$\begin{aligned} gh &= a^k x a^l y \\ &= a^k a^l x y \\ &= a^{k+l} y x \\ &= a^l a^k y x \\ &= a^l y a^k x \\ &= hg \end{aligned}$$

the second equality holding since $x \in Z(G)$, the third since $x, y \in Z(G)$ and the fifth since $y \in Z(G)$. This concludes the proof that G is abelian.

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