HOMEWORK 7: SOLUTIONS - MATH 341
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Problem 1 *(a) Let $G = \langle a \rangle$ be a cyclic group of order 10. Describe explicitly the elements of* $\mathrm{Aut}(G)$.

(b) Determine $\mathrm{Aut}(\mathbf{Z})$.

Solution:

(a) All the homomorphisms of G are completely determined by the image of the generator a. For a homomorphism to be an automorphism it must map a to a generator. Therefore the automorphisms of G are in one-to-one correspondence with the set of its generators:

$$\phi_1 : a \mapsto a, \quad \phi_3 : a \mapsto a^3, \quad \phi_7 : a \mapsto a^7, \quad \phi_9 : a \mapsto a^9.$$

(b) Similarly with (a), since $\mathbf{Z}$ is cyclic, its homomorphisms are completely determined by the image of its generator 1. To be an automorphism, such a homomorphism needs to map the generator 1 to another generator. But 1 and -1 are the only generators of $\mathbf{Z}$. Therefore $\mathbf{Z}$ has only two automorphisms

$$\phi_1 : 1 \mapsto 1 \quad \text{and} \quad \phi_{-1} : 1 \mapsto -1.$$

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Problem 2 *(a) Let G be an Abelian group. Show that the mapping $\phi : G \to G$ defined by letting $\phi(x) = x^{-1}$ for all $x \in G$ is an automorphism of G.*

(b) Show that the mapping $\phi : S_3 \to S_3$ defined by letting $\phi(x) = x^{-1}$ for all $x \in S_3$ is not an automorphism of S_3.

Solution:

(a) We need to show that the given ϕ satisfies the homomorphism condition, that it is 1-1 and onto.

For the homomorphism condition, we have, for all $x, y \in G$,

$$\begin{aligned} \phi(xy) &= (xy)^{-1} \quad \text{(by the definition of } \phi) \\ &= y^{-1}x^{-1} \quad \text{(by a property of inverses)} \\ &= x^{-1}y^{-1} \quad \text{(since } G \text{ is abelian)} \\ &= \phi(x)\phi(y) \quad \text{(by the definition of } \phi). \end{aligned}$$

To show that *phi* is 1-1, let $x, y \in G$. Then, if $\phi(x) = \phi(y)$, we have $x^{-1} = y^{-1}$, whence, by the uniqueness of the inverses in a group, $x = y$.

Finally, suppose that $x \in G$. Then $\phi(x^{-1}) = (x^{-1})^{-1} = x$, whence ϕ is onto and, therefore, it is an automorphism of G.

(b) The key property in (a) was that G be abelian. So, let us consider two non commuting elements of S_3, for instance, (12) and (123). We have $(12)(123) = (23)$, whereas $(123)(12) = (13)$. Hence $\phi((12)(123)) = \phi(23) = (23)^{-1} = (23)$ whereas $\phi((12))\phi((123)) = (12)^{-1}(123)^{-1} = (12)(132) = (13)$.

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Problem 3 *A subgroup H of a group G is called a* characteristic subgroup *of G if for all $\phi \in \mathrm{Aut}(G)$ we have $\phi(H) = H$.*

(a) Show that if H is a characteristic subgroup of G, then $H \lhd G$.

(b) Show that if H is the only subgroup of G of order n, then H is a characteristic subgroup of G.

Solution:

(a) Recall that the mapping $T_g : G \to G; h \mapsto ghg^{-1}$ is an automorphism of G, for every $g \in G$. Automorphisms of this form have been called *inner automorphisms* of G. Since H is a characteristic subgroup of G, $\phi(H) = H$, for all automorphisms of G. Thus, in particular, $T_g(H) = H$, for all inner automorphisms $T_g, g \in G$. Therefore, $gHg^{-1} = H$, for all $g \in G$. But this is one of the conditions of the normal subgroup test. Thus H must be a normal subgroup of G.

(b) Recall that automorphisms preserve the order of a group. So, if H is the only subgroup of G of order n, the subgroup $\phi(H)$ having order n, must be the only subgroup of order n, i.e., H itself. Hence $\phi(H) = H$, for all $\phi \in \mathrm{Aut}(G)$. Therefore H is a characteristic subgroup of G.

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Problem 4 *(a) Show that D_4 and $\mathbf{Z}_2 \times \mathbf{Z}_4$ are not isomorphic.*

(b) In D_4 find a subgroup H such that $H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

Solution:

(a) The two groups have different numbers of elements of orders 2 and 4:

	Order 2	Order 4
D_4	$\rho^2, \tau, \rho\tau, \rho^2\tau, \rho^3\tau$	ρ, ρ^3
$\mathbf{Z}_2 \times \mathbf{Z}_4$	$(0,2), (1,0), (1,2)$	$(0,1), (0,3), (1,1), (1,3)$

(b) The following multiplication tables show that $\{\rho_0, \rho^2, \tau, \rho^2\tau\} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$:

	ρ_0	ρ^2	τ	$\rho^2\tau$
ρ_0	ρ_0	ρ^2	τ	$\rho^2\tau$
ρ^2	ρ^2	ρ_0	$\rho^2\tau$	τ
τ	τ	$\rho^2\tau$	ρ_0	ρ^2
$\rho^2\tau$	$\rho^2\tau$	τ	ρ^2	ρ_0

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

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Problem 5 *Let $H \lhd G_1$ and $K \lhd G_2$. Show that*

(a) $H \times K$ is a subgroup of $G_1 \times G_2$

(b) $H \times K \lhd G_1 \times G_2$

(c) $(G_1 \times G_2)/(H \times K) \cong G_1/H \times G_2/K$

Solution:

(a) Suppose that $(h_1, k_1), (h_2, k_2) \in H \times K$. Then

$$\begin{aligned}(h_1, k_1)(h_2, k_2)^{-1} &= (h_1, k_1)(h_2^{-1}, k_2^{-1}) \\ &= (h_1 h_2^{-1}, k_1 k_2^{-1}) \\ &\in H \times K.\end{aligned}$$

(b) Let $(h, k) \in H \times K$ and $g_1 \in G_1, g_2 \in G_2$. Then, we have

$$\begin{aligned}(g_1, g_2)(h, k)(g_1, g_2)^{-1} &= (g_1, g_2)(h, k)(g_1^{-1}, g_2^{-1}) \\ &= (g_1 h g_1^{-1}, g_2 k g_2^{-1}) \\ &\in H \times K.\end{aligned}$$

(c) Define the homomorphism $\phi : G_1 \times G_2 \to G_1/H \times G_2/K$ by

$$\phi(g_1, g_2) = (g_1 H, g_2 K), \quad \text{for all } (g_1, g_2) \in G_1 \times G_2.$$

It is not difficult to show that ϕ is an onto homomorphism such that $\text{Ker}(\phi) = H \times K$. Therefore, by the first isomorphism theorem, we get

$$(G_1 \times G_2)/(H \times K) \cong G_1/H \times G_2/K.$$

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