HOMEWORK 8: SOLUTIONS - MATH 341
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Problem 1 *(a) Find the order of* (ρ, i) *in the group* $S_3 \times Q_8$.

(b) Find the order of $((2\ 3\ 4), 15)$ *in* $A_4 \times \mathbf{Z}_{18}$.

Solution:

(a) We have $|(\rho, i)| = \text{lcm}(3,4) = 12$.

(b) Similarly,

$$\begin{aligned} |((234),15)| &= \text{lcm}(|(234)|, |15|) \\ &= \text{lcm}(3, \tfrac{18}{\gcd(15,18)}) \\ &= \text{lcm}(3, \tfrac{18}{3}) \\ &= \text{lcm}(3,6) \\ &= 6. \end{aligned}$$

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Problem 2 *(a) Find the distinct cosets of* $H = \langle (3,5) \rangle$ *in the group* $U(10) \times U(12)$.

(b) Find the order of $(3,2) + \langle (6,8) \rangle$ *in* $(3\mathbf{Z} \times 2\mathbf{Z})/\langle (6,8) \rangle$.

Solution:

(a) Recall that $U(10) = \{1,3,7,9\}$ and $U(12) = \{1,5,7,11\}$. Now $H = \langle (3,5) \rangle = \{(3,5),(9,1),(7,5),(1,1)\}$, whence $[U(10) \times U(12) : \langle (3,5) \rangle] = \frac{16}{4} = 4$. The cosets are H itself,

$$(1,7)\langle (3,5) \rangle = \{(3,11),(9,7),(7,11),(1,7)\},$$

$$(1,11)\langle (3,5) \rangle = \{(3,7),(9,11),(7,7),(1,11)\}$$

and

$$(3,1)\langle (3,5) \rangle = \{(9,5),(7,1),(1,5),(3,1)\}.$$

(b) We have

$$2(3,2) + \langle (6,8) \rangle = (0,4) + \langle (6,8) \rangle, \quad 3(3,2) + \langle (6,8) \rangle = (3,6) + \langle (6,8) \rangle,$$

$$4(3,2) + \langle (6,8) \rangle = (0,0) + \langle (6,8) \rangle,$$

whence $|(3,2) + \langle (6,8) \rangle| = 4$.

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Problem 3 *(a) Explain why there are no nontrivial proper subgroups* H *and* K *in* $\mathbf{Z}_8$ *such that* $\mathbf{Z}_8 = H \oplus K$.

(b) Find nontrivial proper subgroups H and K in $U(12)$ such that $HK = U(12)$.

Solution:

(a) By order considerations we must have $|H| = 2$ and $|K| = 4$ (or $|H| = 4$ and $|K| = 2$). Thus $|H| = \langle 4 \rangle$ and $K = \langle 2 \rangle$. But then, the group $H \oplus K$ does not have any element of order 8 and thus $\mathbf{Z}_8 \neq H \oplus K$.

(b) Set $H = \langle 5 \rangle = \{1, 5\}$ and $K = \langle 7 \rangle = \{1, 7\}$. Then $HK = \{1, 5\}\{1, 7\} = \{1, 5, 7, 11\}$.

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Problem 4 *(a) Let H and K be subgroups of a group G such that $G = H \oplus K$, H is cyclic of order 6, and K is cyclic of order 15. Show that G is an Abelian group of order 90 that is not cyclic.*

(b) Let $G = H_1 \oplus \ldots \oplus H_n$, and let $x = h_1 + \ldots + h_n \in G$. Show that $|x| = \mathrm{lcm}(|h_1|, \ldots, |h_n|)$.

Solution:

(a) Let $G = H \oplus K$, with $H = \langle a \rangle, K = \langle b \rangle$, such that $|a| = 6$ and $|b| = 15$. We then have that $G \cong H \times K$, whence G is abelian since both H and K are and $|G| = 6 \times 15 = 90$. G is not cyclic since $\max_{g \in G} |g| = \mathrm{lcm}(|a|, |b|) = \mathrm{lcm}(6, 15) = 30$.

(b) This is a straightforward consequence of the fact that $G \cong H_1 \times \ldots \times H_n$ and that, under this isomorphism, x corresponds to the element $(h_1, h_2, \ldots, h_n)$ of the direct product.

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Problem 5 *Let H and K be subgroups of an Abelian group G and let $\phi : G \to H$ be a homomorphism such that*

(1) $\phi(h) = h$ *for all* $h \in H$

(2) $\mathrm{Kern}(\phi) = K$.

Show that $G = H \oplus K$.

Solution:

Since G is abelian, we have that $H \lhd G$ and $K \lhd G$. So it suffices to show that $H \cap K = \{1\}$ and that $HK = G$.

For the first property, suppose that $x \in H \cap K$. Then $x = \phi(x) = 1$. Therefore $H \cap K = \{1\}$.

For the second property, if $g \in G$, then $\phi(g) \in H$, whence $\phi(\phi(g)) = \phi(g)$, i.e., $\phi(\phi(g))\phi(g)^{-1} = 1$ or $\phi(\phi(g)g^{-1}) = 1$. Therefore $\phi(g)g^{-1} = k$, for some $k \in K$. Thus $g = k^{-1}\phi(g) = \phi(g)k^{-1} \in HK$, as required. ■