EXAM 1: SOLUTIONS - MATH 490
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Problem 1 1. *Give the definitions of a* **one-one** *and of an* **onto** *function.*

2. *Let $f : A \to B$ be a given one-one function and let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of subsets of A. Prove that $f(\cap_{\alpha \in I} X_\alpha) = \cap_{\alpha \in I} f(X_\alpha)$*

Solution:

(a) A function $f : A \to B$ is **one-one** if, for all $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$.

A function $f : A \to B$ is **onto** if, for all $b \in B$, there exists an $a \in A$, such that $f(a) = b$. An alternative way to express this is to say that f is onto if and only if $f(A) = B$.

(b) We show first that $f(\cap_{\alpha \in I} X_\alpha) \subseteq \cap_{\alpha \in I} f(X_\alpha)$. This part does not require that f be one-one. Suppose $y \in f(\cap_{\alpha \in I} X_\alpha)$. Thus, there exists $x \in \cap_{\alpha \in I} X_\alpha)$, such that $y = f(x)$. But $x \in \cap_{\alpha \in I} X_\alpha$ if and only if $x \in X_\alpha$, for all $\alpha \in I$, whence, $y = f(x)$, implies that $y \in f(X_\alpha)$, for all $\alpha \in I$, i.e., $y \in \cap_{\alpha \in I} f(X_\alpha)$. This shows that $f(\cap_{\alpha \in I} X_\alpha) \subseteq \cap_{\alpha \in I} f(X_\alpha)$.

For the reverse inclusion, suppose that $y \in \cap_{\alpha \in I} f(X_\alpha)$. Thus $y \in f(X_\alpha)$, for all $\alpha \in I$. Hence, there exist $x_\alpha \in X_\alpha$, such that $f(x_\alpha) = y$, for all $\alpha \in I$. But f is one-one, whence $f(x_\alpha) = y = f(x_\beta)$ implies $x_\alpha = x_\beta$, for all $\alpha, \beta \in I$. That is $x_\alpha \in X_\beta$, for all $\beta \in I$, i.e., $x_\alpha \in \cap_{\alpha \in I} X_\alpha$, which shows that $y = f(x_\alpha) \in f(\cap_{\alpha \in I} X_\alpha)$, and concludes the proof that $\cap_{\alpha \in I} f(X_\alpha) \subseteq f(\cap_{\alpha \in I} X_\alpha)$.

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Problem 2 1. *Give the definition of a* **relation on** *a set A. Also give the definition of an* **equivalence relation** *on A.*

2. *Let X be the set of functions from the real numbers into the real numbers possessing continuous derivatives. Let R be the subset of $X \times X$ consisting of those pairs (f, g) such that $Df = Dg$ where D maps a function into its derivative. Prove that R is an equivalence relation and describe an equivalence class $\pi(f)$.*

Solution:

(a) A **relation R on** a set A is a subset of the cartesian product $A \times A$, i.e., $R \subseteq A \times A$. A relation R is an equivalence relation if and only if it is reflexive, symmetric and transitive, i.e., if

- $(x, x) \in R$, for all $x \in A$,
- $(x, y) \in R$ implies $(y, x) \in R$, for all $x, y \in A$,

- $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$, for all $x, y, z \in A$.

(b) We first show that R is an equivalence relation: For all $f \in X$, $Df = Df$, whence $(f, f) \in R$ and R is reflexive. For all $f, g \in X$ with $(f, g) \in R$, we have $Df = Dg$, whence $Dg = Df$ and, therefore, $(g, f) \in R$ and R is symmetric. Finally, suppose $(f, g) \in R$ and $(g, h) \in R$. We have $Df = Dg$ and $Dg = Dh$, whence $Df = Dh$ and, therefore, $(f, h) \in R$ and R is transitive. Thus, R is an equivalence relation.

Let $f \in X$. Then we have

$$\begin{aligned} \pi(f) &= \{g \in X : (g, f) \in R\} \\ &= \{g \in X : Dg = Df\} \\ &= \{g \in X : g = f + c, \text{ for some } c \in \mathbb{R}\} \\ &= \{f + c : c \in \mathbb{R}\}. \end{aligned}$$

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Problem 3 *1. Give the definition of a* **metric space***.*

2. Let (X, d) be a metric space. Let k be a positive real number and set $d_k(x, y) = k \cdot d(x, y)$. Prove that (X, d_k) is a metric space.

Solution:

(a) A pair of objects $\langle X, d\rangle$ consisting of a nonempty set X and a function $d : X \times X \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, is called a **metric space** provided that

  (a) $d(x, y) \geq 0$, for all $x, y \in X$,

  (b) $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$,

  (c) $d(x, y) = d(y, x)$, for all $x, y \in X$,

  (d) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

d is the **distance function** or **metric** and X is the **underlying set** of the metric space $\langle X, d\rangle$.

(b) We verify that d_k satisfies all the axioms listed above.

  (a) $d_k(x, y) = kd(x, y) \geq 0$, for all $x, y \in X$, since $k > 0$ and $d(x, y) \geq 0$, for all $x, y \in X$.

  (b) We have $d_k(x, y) = 0$ if and only if $kd(x, y) = 0$ if and only if $d(x, y) = 0$, since $k > 0$, if and only if $x = y$, since d is a metric on X.

  (c) $d_k(x, y) = kd(x, y) = kd(y, x) = d_k(y, x)$, for all $x, y \in X$.

  (d) Finally, $d_k(x, z) = kd(x, z) \leq k(d(x, y) + d(y, z)) = kd(x, y) + kd(y, z) = d_k(x, y) + d_k(y, z)$, for all $x, y \in X$.

This completes the proof that d_k is a metric on X.

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Problem 4 *1. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = x_1 + x_2$. Prove that f is continuous, where the distance function on $\mathbb{R}^2$ is*

$$d((x_1, x_2), (y_1, y_2)) = \max_{i=1,2}\{|x_i - y_i|\}.$$

2. Let (X, d) be a metric space. Define a distance function d^* on $X \times X$ by

$$d^*(x, y) = \max_i\{d(x_i, y_i)\}.$$

Prove that the function $d : (X \times X, d^*) \to (\mathbb{R}, d)$ is continuous.

Solution:

(a) Let $(a_1, a_2) \in \mathbb{R}^2$. Also let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{2} > 0$. Then we have, for all $(x_1, x_2) \in \mathbb{R}^2$, $d((x_1, x_2), (a_1, a_2)) < \delta = \frac{\epsilon}{2}$ implies $\max_{i=1,2}\{|x_i - a_i|\} < \frac{\epsilon}{2}$, whence $|x_i - a_i| < \frac{\epsilon}{2}, i = 1, 2$. Therefore

$$\begin{aligned}
|f(x_1, x_2) - f(a_1, a_2)| &= |x_1 + x_2 - a_1 - a_2| \\
&= |(x_1 - a_1) + (x_2 - a_2)| \\
&\leq |x_1 - a_1| + |x_2 - a_2| \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, f is continuous at all $(a_1, a_2) \in \mathbb{R}^2$.

(b) Let $(a_1, a_2) \in X^2$. Also let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{2} > 0$. Then we have, for all $(x_1, x_2) \in X^2$, $d^*((x_1, x_2), (a_1, a_2)) < \delta = \frac{\epsilon}{2}$ implies $\max_{i=1,2}\{d(x_i, a_i)\} < \frac{\epsilon}{2}$, whence $d(x_i, a_i) < \frac{\epsilon}{2}, i = 1, 2$. Therefore, if $d(x_1, x_2) - d(a_1, a_2) \geq 0$,

$$\begin{aligned}
|d(x_1, x_2) - d(a_1, a_2)| &= d(x_1, x_2) - d(a_1, a_2) \\
&\leq d(x_1, a_1) + d(a_1, a_2) + d(a_2, x_2) - d(a_1, a_2) \\
&= d(x_1, a_1) + d(a_2, x_2) \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

and, similarly, if $d(x_1, x_2) - d(a_1, a_2) < 0$,

$$\begin{aligned}
|d(x_1, x_2) - d(a_1, a_2)| &= -d(x_1, x_2) + d(a_1, a_2) \\
&\leq -d(x_1, x_2) + d(a_1, x_1) + d(x_1, x_2) - d(x_2, a_2) \\
&= d(a_1, x_1) + d(x_2, a_2) \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, f is continuous at all $(a_1, a_2) \in X^2$.

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Problem 5 *Let (X, d_1) and (Y, d_2) be metric spaces. Let $f : X \to Y$ be continuous. Define a distance function d on $X \times Y$ in the standard manner. Prove that the graph Γ_f of f is a closed subset of $(X \times Y, d)$.*

Solution:

We show that $\Gamma(f) \subseteq X \times Y$ is closed in $(X \times Y, d)$ by showing that the limit of every converging sequence of points in $\Gamma(f)$ belongs to $\Gamma(f)$.

So suppose that $(x_n, y_n) \in \Gamma(f), n = 1, 2, \ldots,$ with $(x_n, y_n) \longrightarrow (x, y) \in X \times Y$. Then, by the definition of $\Gamma(f)$, $y_n = f(x_n), n = 1, 2, \ldots,$ and, by a convergence criterion for the product space $x_n \longrightarrow x$ and $y_n \longrightarrow y$. Therefore we have

$$\begin{aligned} y &= \lim y_n \\ &= \lim f(x_n) \\ &= f(\lim x_n) \\ &= f(x), \end{aligned}$$

which shows that $(x, y) \in \Gamma(f)$ and, therefore, $\Gamma(f)$ is closed in $X \times Y$. ■