EXAM 2: SOLUTIONS - MATH 490
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Problem 1 1. *Give the definition of a* **topological space**.

2. *Let X be an arbitrary set. Let $\mathcal{T}$ be the collection of all subsets of X whose complements are either finite or all of X. Then $(X, \mathcal{T})$ is a topological space.*

Solution:

(a) Let X be a non-empty set and $\mathcal{T}$ a collection of subsets of X such that

(O1) $X \in \mathcal{T}$

(O2) $\emptyset \in \mathcal{T}$

(O3) If $O_1, \ldots, O_n \in \mathcal{T}$, then $O_1 \cap \ldots \cap O_n \in \mathcal{T}$

(O4) If, for each $\alpha \in I, O_\alpha \in \mathcal{T}$, then $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{T}$.

The pair of objects $(X, \mathcal{T})$ is called a *topological space*. The set X is called the *underlying set*, the collection $\mathcal{T}$ is called the *topology* on X and the members of $\mathcal{T}$ are the *open sets*.

(b) We need to verify the four conditions above for the given collection $\mathcal{T}$.

We have $X - X = \emptyset$ and $X - \emptyset = X$, whence $X, \emptyset \in \mathcal{T}$.

Now suppose that $O_1, \ldots, O_n \in \mathcal{T}$. Then $X - O_1, X - O_2, \ldots, X - O_n$ are either finite or all of X. Then $X - \bigcap_{i=1}^n O_i = \bigcup_{i=1}^n X - O_i$ is finite if all of $X - O_i$ are finite and it is all of X if at least one of the $X - O_i$ is all of X. In either case $\bigcup_{i=1}^n O_i \in \mathcal{T}$.

Finally, assume that, for all $\alpha \in I, O_\alpha \in \mathcal{T}$. Then $X - O_\alpha$ is either finite or all of X, for all $\alpha \in I$. Then $X - \bigcup_{\alpha \in I} O_\alpha = \bigcap_{\alpha \in I} X - O_\alpha$ is finite if at least one of $X - O_\alpha$ is finite and it is all of X if $X - O_\alpha = X$, for all $\alpha \in I$. In either case $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{T}$. Thus $\mathcal{T}$ is a topology on X.

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Problem 2 1. *Give the definition of a* **metrizable** *topological space.*

2. *Prove that for each set X, the topological space $(X, 2^X)$ is metrizable.*

Solution:

(a) A topological space is said to be *metrizable* if its topology is induced by a metric, i.e., if its topology is the collection of the open sets of an appropriate metric space on its underlying set.

(b) Define a metric $d : X \times X \to \mathbb{R}$ on X by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{otherwise} \end{cases}$$

For all $x \in X$, $\{x\} = B(x; \frac{1}{2})$, whence all singletons are open sets in this metric space. But then, for all $A \subset X$, $A = \bigcup_{a \in A} \{a\}$, whence A is open. Thus every set is an open set and, consequently, the induced topological space is $(X, 2^X)$.

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Problem 3 *1. Give the definition of a* **neighborhood space***.*

2. Given a real number x, call a subset N of $\mathbb{R}$ a neighborhood of x if $y \geq x$ implies $y \in N$. Prove that this definition of neighborhood yields a neighborhood space. Describe the corresponding topological space.

Solution:

(a) Let X be a set. For each $x \in X$, let there be given a collection $\mathcal{N}_x$ of subsets of X satisfying the conditions

N1. For each $x \in X, \mathcal{N}_x \neq \emptyset$;

N2. For each $x \in X$ and $N \in \mathcal{N}_x, x \in N$;

N3. For each $x \in X$ and $N \in \mathcal{N}_x$, if $N' \supset N$ then $N' \in \mathcal{N}_x$;

N4. For each $x \in X$ and $M, N \in \mathcal{N}_x, N \cap M \in \mathcal{N}_x$;

N5. For each $x \in X$ and $N \in \mathcal{N}_x$, there exists an $O \in \mathcal{N}_x$ such that $O \subset N$ and $O \in \mathcal{N}_y$ for each $y \in O$.

This object is called a *neighborhood space*.

(b) We check that the given space satisfies axioms N1-N5, given above. Clearly, $[x, \infty) \in \mathcal{N}_x$, whence $\mathcal{N}_x \neq \emptyset$, for all $x \in \mathbb{R}$ and N1 is satisfied. Since $x \geq x, x \in N$, for all $N \in \mathcal{N}_x$ and N2 is also satisfied. Now suppose that $x \in \mathbb{R}$ and $N' \supset N \in \mathcal{N}_x$. Then $y \geq x$ implies $y \in N \subseteq N'$, whence $y \in N'$ and $N' \in \mathcal{N}_x$. So N3 is satisfied. For N4, if $N, M \in \mathcal{N}_x$ and $y \geq x$, then $y \in N$ and $y \in M$, whence $y \in N \cap M$ and $N \cap M \in \mathcal{N}_x$ as well. Finally, let $x \in \mathbb{R}$ and $N \in \mathcal{N}_x$. Consider the set $O = [x, \infty)$. Clearly $O \in \mathcal{N}_x$ and $O \subseteq N$, since, if $y \in O$, then $y \geq x$ and $y \in N$. Finally, if $z \in O$, then $w \in O$, for all $w \geq z$, whence $O \in \mathcal{N}_z$, for all $z \in O$. Thus, N5 is also satisfied.

Since a subset of $\mathbb{R}$ is open in the induced topology if and only if it is a neighborhood of each of its points, the open subsets of this topology are the subsets $\mathcal{T} = \{[x, \infty) : x \in \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$.

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Problem 4 *1. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Give the definitions of the* **closure, interior** *and* **boundary** *of A.*

2. In $\mathbb{R}^3$ with the usual topology, let A be the set of points $x = (x_1, x_2, x_3)$ such that $x_3 = 0$. Prove that $\mathrm{Int}(A) = \emptyset, \mathrm{Bdry}(A) = A$ *and* $\bar{A} = A$.

Solution:

(a) Let A be a subset of a topological space. A point x is said to be *in the closure* of A if, for each neighborhood N of x, $N \cap A \neq \emptyset$. x is said to be *in the interior* of A if A is a neighborhood of x. Finally, x is *in the boundary* of A if x is both in the closure of A and in the closure of the complement of A.

(b) Suppose that $x \in \mathrm{Int}(A)$. Then $x = (x_1, x_2, 0)$ and there exists $\epsilon > 0$, such that $B(x, \epsilon) \subseteq A$. But $(x_1, x_2, \frac{\epsilon}{2}) \in B(x, \epsilon)$ and $(x_1, x_2, \frac{\epsilon}{2}) \notin A$, which contradicts the inclusion above. Therefore $\mathrm{Int}(A) = \emptyset$.

Obviously, $A \subseteq \overline{A}$. Conversely, if $x \notin A$, then $x = (x_1, x_2, x_3)$, with $x_3 \neq 0$. Therefore, the ball $B(x, \frac{|x_3|}{2})$ is a neighborhood of x that does not intersect A. Hence $\overline{A} \subseteq A$ which yields that $\overline{A} = A$.

One shows similarly that $\overline{C(A)} = \mathbb{R}^3$, whence $\overline{A} \cap \overline{C(A)} = \mathbb{R}^3 \cap A = A$. Thus $\mathrm{Bdry}(A) = A$.

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Problem 5 *1. Give the definition of a* **Hausdorff space**.

2. Prove that a subspace of a Hausdorff space is a Hausdorff space.

Solution:

(a) A topological space $(X, \mathcal{T})$ is called a *Hausdorff space* or is said to satisfy the *Hausdorff axiom* if for each pair a, b of distinct points of X there are neighborhoods N and M of a and b respectively, such that $N \cap M = \emptyset$.

(b) Let X be a Hausdorff space and let Y be a subspace of X. Consider $a, b \in Y$, with $a \neq b$. Then, since X is Hausdorff, there exist neighborhoods N and M of a and b respectively, such that $N \cap M = \emptyset$. Consider the relative neighborhoods $N' = N \cap Y$ and $M' = M \cap Y$ of a and b in Y, respectively. Since $N \cap M = \emptyset$, we also have $N' \cap M' = \emptyset$, whence Y is also Hausdorff.

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