EXAM 3: SOLUTIONS - MATH 490
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Problem 1 *1. Give the definition of the* **product** *of finitely many topological spaces.*

2. Prove that a subset F of $X = \prod_{i=1}^n X_i$ is closed if and only if F is an intersection of sets , each of which is a finite union of sets of the form $F_1 \times \cdots \times F_n$, where each F_i is a closed subset of X_i.

Solution:

(a) The topological space $\langle X, \mathcal{T}\rangle$, where $\mathcal{T}$ is the collection of all subsets of X that are unions of sets of the form $O_1 \times \ldots \times O_n$, each O_i an open subset of X_i, is called the *product* of the topological spaces $\langle X_i, \mathcal{T}_i\rangle, i = 1, 2, \ldots, n$.

(b) F is closed in $X = \prod_{i=1}^n X_i$ if and only if $X - F$ is open in X if and only if $X - F = \bigcup_{j\in J}(O_1^j \times \ldots \times O_n^j)$ for open $O_i^j, j \in J$, in $X_i, i = 1, \ldots, n$, if and only if $F = X - \bigcup_{j\in J}(O_1^j \times \ldots \times O_n^j)$ if and only if $F = \bigcap_{j\in J}(X - O_1^j \times \ldots \times O_n^j)$ if and only if $F = \bigcap_{j\in J}\bigcup_{i=1}^n(X_1 \times \ldots \times X_{i-1} \times (X_i - O_i^j) \times X_{i+1} \times \ldots \times X_n)$. Note that $X_1 \times \ldots \times X_{i-1} \times (X_i - O_i^j) \times X_{i+1} \times \ldots \times X_n$ is closed in X, whence the conclusion follows.

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Problem 2 *1. Give the definition of a* **category**.

2. Let C be an arbitrary category and X an object in C. Verify that the set of isomorphisms in $H(X, X)$ is a group under the operation of category composition.

Solution:

(a) A *category* C is a collection of objects A whose members are called the *objects* of the category and for each ordered pair (X, Y) of objects of the category a set $H(X, Y)$ called the *maps* of X into Y together with a rule of composition which associates to each $f \in H(X, Y)$ and $g \in H(Y, Z)$ a map $gf \in H(X, Z)$. This composition is associative, that is, if $f \in H(X, Y), g \in H(Y, Z), h \in H(Z, W)$, then $h(gf) = (hg)f$ and identities exist, that is, for each object $X \in A$ there is an element $1_X \in H(X, X)$ such that for all $g \in H(X, Y), g1_X = g$ and for all $h \in H(W, X), 1_X h = h$.

(b) Let $f, g, h \in H(X, X)$. Then $h(gf) = (hg)f$, by the associativity of the category composition. 1_X acts as an identity in $H(X, X)$ under composition, by the identity category axiom, and, for every f, since f is an isomorphism, there exists an f^{-1}, such that $ff^{-1} = f^{-1}f = 1_X$. Therefore the isomorphisms of $H(X, X)$ form a group under composition.

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Problem 3 *1. Give the definition of a* **connected topological space***.*

2. Show that the set $\mathbf{Q}$ *of the rational numbers with the subspace topology inherited from the usual topology of the real numbers is disconnected.*

Solution:

(a) A topological space is said to be *connected* if the only two subsets of X that are simultaneously open and closed are X itself and the empty set $\emptyset$.

(b) Consider the following two subsets of $\mathbf{Q}$: $(-\infty, \sqrt{2}) \cap \mathbf{Q}$ and $(\sqrt{2}, +\infty) \cap \mathbf{Q}$. These are two nonempty disjoint relatively open subsets of $\mathbf{Q}$ whose union is all of $\mathbf{Q}$. Therefore $\mathbf{Q}$ is disconnected.

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Problem 4 *1. Give a function f from a closed interval of real numbers with the usual topology into the set of real numbers, also with the usual topology, with a connected graph that is not continuous.*

2. Let X be the set of real numbers with the topology $\mathcal{T} = \{U \subset X : 0 \in U\} \cup \{\emptyset\}$. Is the space $(X, \mathcal{T})$ connected? How about the subspace $X - \{0\}$?

Solution:

(a) Define $f : [0,1] \to \mathrm{I\!R}$, by $f(x) = \sin \frac{1}{x}$, if $x \neq 0$, and $f(0) = 0$. Then the graph of f, $G_f = \{(x, f(x)) : x \in [0,1]\}$, is a connected subset of the plane with the usual topology, since $(0,0) \in \overline{G_f - \{(0,0)\}}$ and $G_f - \{(0,0)\}$ is obviously connected, but f is not continuous at $x = 0$.

(b) $\langle X, \mathcal{T} \rangle$ is connected, since, if P, Q are nonempty open subsets of $\langle X, \mathcal{T} \rangle$, $0 \in P \cap Q \neq \emptyset$.

$X - \{0\}$ is not connected, however, since $P = (-\infty, 0) = (-\infty, 0] \cap (X - \{0\})$ and $Q = (0, \infty) = [0, \infty) \cap (X - \{0\})$ are two nonempty disjoint relatively open subsets of $X - \{0\}$ such that $P \cup Q = X - \{0\}$.

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Problem 5 *1. Give the definition of a* **path-connected space***.*

2. Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed family of topological spaces and set $X = \prod_{\alpha \in A} X_\alpha$. For each $\alpha \in A$ let $f_\alpha : I \to X_\alpha$ be a path in X_α. Set $(f_A(t))(\alpha) = f_\alpha(t)$ so that $f_A : I \to X$. Prove that f_A is a path in X. Prove that if each X_α is path-connected, so is X.

Solution:

(a) A topological space is *path-connected* if, for each pair of points $u, v \in X$, there is a path f connecting u to v.

(b) Since every projection $(p_\alpha f_A(t) = p_\alpha(f_A(t)) = f_\alpha(t)$ is continuous, the function $f_A : I \to X$ is continuous.

Let $x = (x_\alpha)_{\alpha \in I}, y = (y_\alpha)_{\alpha \in I} \in X$. Since X_α is path-connected, there exists $f_\alpha : I \to X_\alpha$, such that $f_\alpha(0) = x_\alpha$ and $f_\alpha(1) = y_\alpha$, for all $\alpha \in I$. Then $f_A : I \to X$, with $(f_A(t))(\alpha) = f_\alpha(t)$ is a path in X, such that $f_A(0) = x$ and $f_A(1) = y$. Therefore X is also path-connected.

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