HOMEWORK 1 - MATH 490

DUE DATE: Monday, January, 27
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work.

GOOD LUCK!!

1. Let $X \subset A, Y \subset B$. Prove that $C(X \times Y) = A \times C(Y) \cup C(X) \times B$.

2. Let $f : A \to B$ be given and let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of subsets of A. Prove that
   (a) $f(\cup_{\alpha \in I} X_\alpha) = \cup_{\alpha \in I} f(X_\alpha)$
   (b) $f(\cap_{\alpha \in I} X_\alpha) \subset \cap_{\alpha \in I} f(X_\alpha)$
   (c) If $f : A \to B$ is one-one, then $f(\cap_{\alpha \in I} X_\alpha) = \cap_{\alpha \in I} f(X_\alpha)$.

3. Let $f : X \to Y$ be a function from a set X onto a set Y. Let R be the subset of $X \times X$ consisting of those pairs (x, x'), such that $f(x) = f(x')$. Prove that R is an equivalence relation. Let $\pi : X \to X/R$ be the projection. Verify that, if $\alpha \in X/R$ is an equivalence class, to define $F(\alpha) = f(a)$, whenever $\alpha = \pi(a)$, establishes a well-defined function $F : X/R \to Y$ which is one-one and onto.

4. Let $\mathbb{R}$ be the real numbers and ∞ an object not in $\mathbb{R}$. Define a set $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Let a, b, c, d be real numbers. Let $f : \mathbb{R}^* \to \mathbb{R}^*$ be a function defined by $f(x) = \frac{ax+b}{cx+d}$ when $x \neq -\frac{d}{c}, \infty$ while $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$. Prove that f has an inverse provided that $ad - bc \neq 0$.

5. Let m, n be positive integers. Let X be a set with m distinct elements and Y a set with n distinct elements. How many distinct functions are there from X to Y? Let A be a subset of X with r distinct elements, $0 \leq r < m$ and $f : A \to Y$. How many distinct extensions of f to X are there?

6. Let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of sets and let $I = I_1 \cup I_2$, where $I_1 \cap I_2 = \emptyset$. Show that there is a one-one mapping of $(\prod_{\alpha \in I_1} X_\alpha) \times (\prod_{\alpha \in I_2} X_\alpha)$ onto $\prod_{\alpha \in I} X_\alpha$.

7. Prove that $(\mathbb{R}^n, d'')$ is a metric space, where the function d'' is defined by the correspondence

$$d''(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. In $(\mathbb{R}^2, d'')$ determine the shape and position of the set of points x, such that $d''(x, a) \leq 1$ for a specific point $a \in \mathbb{R}^2$.

8. (a) Let X be the set of all continuous functions $f : [a, b] \to \mathbb{R}$. For $f, g \in X$, define

$$d(f, g) = \int_a^b |f(t) - g(t)| dt.$$

Using appropriate theorems from calculus, prove that (X, d) is a metric space.

(b) Let X be a set. For $x, y \in X$ define the function d by

$$d(x, x) = 0, \quad \text{and} \quad d(x, y) = 1, \text{ if } x \neq y,$$

Prove that (X, d) is a metric space.