HOMEWORK 1: SOLUTIONS - MATH 490
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Problem 1 *Let $X \subset A, Y \subset B$. Prove that $C(X \times Y) = A \times C(Y) \cup C(X) \times B$.*

Solution: We show first that $C(X \times Y) \subseteq A \times C(Y) \cup C(X) \times B$. Let $(a,b) \in C(X \times Y)$. Then $(a,b) \notin X \times Y$. Therefore $a \notin X$ or $b \notin Y$. So we study these two cases:

If $a \notin X$, then $a \in C(X)$. Since $b \in B$, we then have $(a,b) \in C(X) \times B$, whence (a,b) $A \times C(Y) \cup C(X) \times B$, as was to be shown.

If, on the other hand, $b \notin Y$, then $b \in C(Y)$, whence, since $a \in A$, $(a,b) \in A \times C(Y)$ and, therefore, $(a,b) \in A \times C(Y) \cup C(X) \times B$, as well.

Therefore, in both cases, $(a,b) \in A \times C(Y) \cup C(X) \times B$, and we have $C(X \times Y) \subseteq A \times C(Y) \cup C(X) \times B$.

Conversely, we need to show that $A \times C(Y) \cup C(X) \times B \subseteq C(X \times Y)$. So suppose that $(a,b) \in A \times C(Y) \cup C(X) \times B$. Then $(a,b) \in A \times C(Y)$ or $(a,b) \in C(X) \times B$. We again consider these two cases separately.

If $(a,b) \in A \times C(Y)$, then $a \in A$ and $b \in C(Y)$, i.e., $a \in A$ and $b \notin Y$, whence $(a,b) \notin X \times Y$ and, therefore $(a,b) \in C(X \times Y)$.

If, on the other hand, $(a,b) \in C(X) \times B$, then $a \in C(X)$ and $b \in B$, whence $a \notin X$ and $b \in B$. Thus, $(a,b) \notin X \times Y$, which shows that $(a,b) \in C(X \times Y)$.

Thus, in both cases, $(a,b) \in C(X \times Y)$ and we have that $A \times C(Y) \cup C(X) \times B \subseteq C(X \times Y)$. ■

Problem 2 *Let $f : A \to B$ be given and let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of subsets of A. Prove that*

(a) $f(\cup_{\alpha \in I} X_\alpha) = \cup_{\alpha \in I} f(X_\alpha)$

(b) $f(\cap_{\alpha \in I} X_\alpha) \subset \cap_{\alpha \in I} f(X_\alpha)$

(c) If $f : A \to B$ is one-one, then $f(\cap_{\alpha \in I} X_\alpha) = \cap_{\alpha \in I} f(X_\alpha)$.

Solution:

(a)

$$\begin{array}{rll} b \in f(\cup_{\alpha \in I} X_\alpha) & \text{iff} & \exists a \in \cup_{\alpha \in I} X_\alpha : b = f(a) \\ & \text{iff} & \exists \alpha \in I \exists a \in X_\alpha : b = f(a) \\ & \text{iff} & \exists \alpha \in I : b \in f(X_\alpha) \\ & \text{iff} & b \in \cup_{\alpha \in I} f(X_\alpha) \end{array}$$

(b) Suppose that $b \in f(\cap_{\alpha \in I} X_\alpha)$. Then, there exists an $a \in \cap_{\alpha \in I} X_\alpha$, such that $b = f(a)$. But then $a \in X_\alpha$ for all $\alpha \in I$ and $b = f(a)$. Therefore $b \in f(X_\alpha)$, for all $\alpha \in I$. Thus $b \in \cap_{\alpha \in I} f(X_\alpha)$. Thus, we have shown that $f(\cap_{\alpha \in I} X_\alpha) \subset \cap_{\alpha \in I} f(X_\alpha)$.

(c) Note that, by part (b), $f(\cap_{\alpha\in I}X_\alpha) \subset \cap_{\alpha\in I}f(X_\alpha)$, even without the assumption that f is one-one.

For the reverse inclusion, suppose that $f : A \to B$ is one-one and let $a \in \cap_{\alpha\in I}f(X_\alpha)$. Then, $a \in f(X_\alpha)$, for all $\alpha \in I$. Thus, for all $\alpha \in I$, there exists $a_\alpha \in X_\alpha$, such that $b = f(a_\alpha)$. Since $b = f(a_\alpha)$, for all $\alpha \in I$, $a_\alpha \in X_\alpha \subseteq A$, and $f : A \to B$ is one-one, it follows that $a_\alpha = a_{\alpha'}$, for all $\alpha, \alpha' \in I$. hence $a_\alpha \in \cap_{\alpha\in I}X_\alpha$, which, together with $b = f(a_\alpha)$, shows that $b \in f(\cap_{\alpha\in I}X_\alpha)$. Therefore $\cap_{\alpha\in I}f(X_\alpha) \subset f(\cap_{\alpha\in I}X_\alpha)$ if f is one-one. ■

Problem 3 *Let $f : X \to Y$ be a function from a set X onto a set Y. Let R be the subset of $X \times X$ consisting of those pairs (x, x'), such that $f(x) = f(x')$. Prove that R is an equivalence relation. Let $\pi : X \to X/R$ be the projection. Verify that, if $\alpha \in X/R$ is an equivalence class, to define $F(\alpha) = f(a)$, whenever $\alpha = \pi(a)$, establishes a well-defined function $F : X/R \to Y$ which is one-one and onto.*

Solution: We have $f(x) = f(x)$, whence xRx and R is reflexive. If xRy, then $f(x) = f(y)$, whence $f(y) = f(x)$, i.e., yRx and R is symmetric. Finally, if xRy and yRz, then $f(x) = f(y)$ and $f(y) = f(z)$, whence $f(x) = f(z)$ and xRz. Thus R is also transitive, i.e., it is an equivalence relation on X.

First, we show that F is well-defined. Suppose, for this purpose, that $x, y \in X$, such that xRy. We need to show that $F(x) = F(y)$. Since xRy, we have $f(x) = f(y)$, by the definition of R, whence $F(x) = f(x) = f(y) = F(y)$, by the definition of F, and F is well-defined.

Next, F is onto, since, for $y \in Y$, there exists an $x \in X$, such that $y = f(x)$, by the fact that f is onto. But then $F(x/R) = f(x) = y$.

Finally, F is one-one, since, if $F(x/R) = F(y/R)$, then $f(x) = f(y)$, whence xRy, i.e., $x/R = y/R$. ■

Problem 4 *Let $\mathbb{R}$ be the real numbers and ∞ an object not in $\mathbb{R}$. Define a set $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Let a, b, c, d be real numbers. Let $f : \mathbb{R}^* \to \mathbb{R}^*$ be a function defined by $f(x) = \frac{ax+b}{cx+d}$ when $x \neq -\frac{d}{c}, \infty$ while $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$. Prove that f has an inverse provided that $ad - bc \neq 0$.*

Solution: Since $ad - bc \neq 0$, not both a and c can be zero. Therefore, we may define the function $g : \mathbb{R}^* \to \mathbb{R}^*$, such that

$$g(x) = \frac{-dx + b}{cx - a}, \quad \text{if } x \neq \frac{a}{c}, \infty,$$

while $g(\frac{a}{c}) = \infty$ and $g(\infty) = -\frac{d}{c}$. Now it is not difficult to check that

$$g \circ f = 1_{\mathbb{R}^*} \quad \text{and} \quad f \circ g = 1_{\mathbb{R}^*}.$$

For instance, for the first equation, for all $x \neq -\frac{d}{c}, \infty$, we have

$$g(f(x)) = g(\frac{ax + b}{cx + d}) = \frac{-d\frac{ax+b}{cx+d} + b}{c\frac{ax+b}{cx+d} - a} = \frac{-dax - db + bcx + bd}{cax + cb - acx - ad} = \frac{(bc - da)x}{bc - da} = x = 1_{\mathbb{R}^*}(x).$$

■

Problem 5 *Let m, n be positive integers. Let X be a set with m distinct elements and Y a set with n distinct elements. How many distinct functions are there from X to Y? Let A be a subset of X with r distinct elements, $0 \leq r < m$ and $f : A \to Y$. How many distinct extensions of f to X are there?*

Solution: Since for each element of X we have n distinct choices for its image under f, the multiplication principle gives n^m distinct functions from X to Y.

In the second case, since the functions have to agree with f on A, there is only one choice for the elements of A, but n different choices for every element in $X - A$. Thus, in this case, there are n^{m-r} distinct extensions of f to X. ■

Problem 6 *Let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of sets and let $I = I_1 \cup I_2$, where $I_1 \cap I_2 = \emptyset$. Show that there is a one-one mapping of $(\prod_{\alpha \in I_1} X_\alpha) \times (\prod_{\alpha \in I_2} X_\alpha)$ onto $\prod_{\alpha \in I} X_\alpha$.*

Solution: Define $F : (\prod_{\alpha \in I_1} X_\alpha) \times (\prod_{\alpha \in I_2} X_\alpha) \to \prod_{\alpha \in I} X_\alpha$ by

$$F(f, g)(\alpha) = \begin{cases} f(\alpha), & \text{if } \alpha \in I_1 \\ g(\alpha), & \text{if } \alpha \in I_2 \end{cases},$$

for all $f \in \prod_{\alpha \in I_1} X_\alpha, g \in \prod_{\alpha \in I_2} X_\alpha$.

It is not difficult to show that F is one-one and onto.

For one-one, suppose that $F(f, g) = F(h, k)$. Then $F(f, g)(\alpha) = F(h, k)(\alpha)$, for all $\alpha \in I_1$, and $F(f, g)(\alpha) = F(h, k)(\alpha)$, for all $\alpha \in I_2$. Therefore, $f(\alpha) = h(\alpha)$, for all $\alpha \in I_1$, and $g(\alpha) = k(\alpha)$, for all $\alpha \in I_2$. Hence $f = h$ and $g = k$, whence $(f, g) = (h, k)$, as needed.

For onto, suppose that $f \in \prod_{\alpha \in I} X_\alpha$. Then, for $f|_{I_1} \in \prod_{\alpha \in I_1} X_\alpha$ and $f|_{I_2} \in \prod_{\alpha \in I_2} X_\alpha$, we have $F(f|_{I_1} \in \prod_{\alpha \in I_1} X_\alpha, f|_{I_2} \in \prod_{\alpha \in I_2} X_\alpha) = f$, and F is onto, as required. ■

Problem 7 *Prove that $(\mathbb{R}^n, d'')$ is a metric space, where the function d'' is defined by the correspondence*

$$d''(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. In $(\mathbb{R}^2, d'')$ determine the shape and position of the set of points x, such that $d''(x, a) \leq 1$ for a specific point $a \in \mathbb{R}^2$.

Solution: For the first property, $|x_i - y_i| \geq 0$, for all $i = 1, \ldots, n$, whence $d''(x, y) = \sum_{i=1}^{n} |x_i - y_i| \geq 0$. For the second property, if $x = y$, $x_i = y_i$, for all $i = 1, \ldots, n$, whence $|x_i - y_i| = 0$, for all $i = 1, \ldots, n$, and, therefore, $d''(x, y) = \sum_{i=1}^{n} |x_i - y_i| = 0$, and, conversely, if $d''(x, y) = \sum_{i=1}^{n} |x_i - y_i| = 0$, then, since, $|x_i - y_i| \geq 0$, for all $i = 1, \ldots, n$, we must have $|x_i - y_i| = 0$, for all $i = 1, \ldots, n$, whence $x_i = y_i$, for all $i = 1, \ldots, n$, i.e., $x = y$. Symmetry is obvious, since $|x_i - y_i| = |y_i - x_i|$, for all $i = 1, \ldots, n$. Finally, for the triangle inequality, we have

$$\begin{array}{rcl} d''(x,z) & = & \sum_{i=1}^n |x_i - z_i| \\ & \leq & \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) \\ & = & \sum_{i=1}^n |x_i - y_i| + \sum_{i=1}^n |y_i - z_i| \\ & = & d''(x,y) + d''(y,z). \end{array}$$

Now, let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then $d''(x,a) \leq 1$, whence $\sum_{i=1}^n |x_i - a_i| \leq 1$. Thus $\{x \in \mathbb{R}^n : d''(x,a) \leq 1\} = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i - a_i| \leq 1\}$, which is an n-cube centered at a with diagonals of length 2 parallel to the coordinate axes. ■

Problem 8 *(a) Let X be the set of all continuous functions $f : [a,b] \to \mathbb{R}$. For $f, g \in X$, define*

$$d(f,g) = \int_a^b |f(t) - g(t)| dt.$$

Using appropriate theorems from calculus, prove that (X, d) is a metric space.

(b) Let X be a set. For $x, y \in X$ define the function d by

$$d(x,x) = 0, \quad \textit{and} \quad d(x,y) = 1, \ \textit{if } x \neq y,$$

Prove that (X, d) is a metric space.

Solution:

(a) Since $|f(t) - g(t)| \geq 0$, for all $t \in [a,b]$, then $\int_a^b |f(t) - g(t)| dt \geq 0$, whence $d(f,g) \geq 0$.

If $f = g$, then $|f(t) - g(t)| = 0$, for all $t \in [a,b]$, whence $d(f,g) = \int_a^b |f(t) - g(t)| dt = 0$. Conversely, if $d(f,g) = \int_a^b |f(t) - g(t)| dt = 0$, then, since f, g are continuous, $|f(t) - g(t)| = 0$, for all $t \in [a,b]$, whence $f(t) = g(t)$, for all $t \in [a,b]$, i.e., $f = g$.

Finally, for the triangle inequality, we have $|f(t) - h(t)| \leq |f(t) - g(t)| + |g(t) - h(t)|$, for all $t \in [a,b]$, whence $\int_a^b |f(t) - h(t)| dt \leq \int_a^b (|f(t) - g(t)| + |g(t) - h(t)|) dt$, whence $\int_a^b |f(t) - h(t)| dt \leq \int_a^b |f(t) - g(t)| dt + \int_a^b |g(t) - h(t)| dt$, and, therefore, $d(f,g) \leq d(f,g) + d(g,h)$.

(b) Clearly, $d(x,y) \geq 0$, for all $x, y \in X$. Also, it is obvious that $d(x,y) = 0$ if and only if $x = y$ and that $d(x,y) = d(y,x)$, for all $x, y \in X$. For the triangle inequality, if $x \neq y$ or $y \neq z$, then $d(x,y) + d(y,z) \geq 1 \geq d(x,z)$ and we are done. The only remaining case is when $x = y$ and $y = z$, whence $x = z$ and we have $d(x,z) = 0 = d(x,y) + d(y,z)$, as required. ■