HOMEWORK 2 - MATH 490

DUE DATE: Monday, February 10
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work.

GOOD LUCK!!

1. Let X be the set of continuous functions $f : [a, b] \to \mathbb{R}$. Let d^* be the distance function on X defined by $d^*(f, g) = \int_a^b |f(t) - g(t)|dt$, for $f, g \in X$. For each $f \in X$, set $I(f) = \int_a^b f(t)dt$. Prove that the function $I : (X, d^*) \to (\mathbb{R}, d)$ is continuous.

2. Let $(X_i, d_i), (Y_i, d_i'), i = 1, 2, \ldots, n$ be metric spaces. Let $f_i : X_i \to Y_i, i = 1, \ldots, n$ be continuous functions. Let $X = \prod_{i=1}^n X_i$ and $Y = \prod_{i=1}^n Y_i$ and convert X and Y into metric spaces in the standard manner. Define the function $F : X \to Y$ by
$$F(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n)).$$
Prove that f is continuous.

3. Let (X, d) be a metric space such that $d(x, y) = 1$ whenever $x \neq y$. Let $a \in X$. Prove that $\{a\}$ is a neighborhood of a and constitutes a basis for the system of neighborhoods at a. Prove that every subset of X is a neighborhood of each of its points.

4. (a) Let a be a point in a metric space X. Let N be the set of positive integers. Prove that there is a collection $\{B_n\}_{n \in N}$ of neighborhoods of a which constitutes a basis for the system of neighborhoods at a.
   (b) Let a and b be distinct points of a metric space X. Prove that there are neighborhoods N_a and N_b of a and b, respectively, such that $N_a \cap N_b = \emptyset$.

5. (a) Let $X_1, X_2, \ldots, X_k$ be metric spaces and convert $X = \prod_{i=1}^k X_i$ into a metric space in the standard manner. Each of the points $a_1, a_2, \ldots$ of a sequence of points of X has k coordinates; that is $a_n = (a_1^n, \ldots, a_k^n) \in X, n = 1, 2, \ldots$. Let $c = (c_1, c_2, \ldots, c_k) \in X$. Prove that $\lim_n a_n = c$ if and only if $\lim_n a_i^n = c_i, i = 1, 2, \ldots, k$.

(b) Prove that a subsequence of a convergent sequence is convergent and converges to the same limit as the original sequence.

6. (a) A sequence of real numbers $a_1, a_2, \ldots$ is called *monotone non-decreasing* if $a_i \leq a_{i+1}$ for each i and called *monotone non-increasing* if $a_i \geq a_{i+1}$ for each i. A sequence which is either monotone non-decreasing or monotone non-increasing is called *monotone*. The sequence is said to be *bounded above* if there is a number K such that $a_i \leq K$ for each i and *bounded below* if there is a number M such that $a_i \geq M$ for each i. A sequence which is both bounded above and bounded below is called *bounded*. Prove that a convergent sequence of real numbers is bounded. Prove that a monotone non-decreasing sequence of real numbers which is bounded above converges to a limit a and that a is the least upper bound of the set $\{a_1, a_2, \ldots\}$.

   (b) Let A be a nonempty subset of a metric space (X, d). Define the function $f : X \to \mathbb{R}$ by $f(x) = d(x, A)$. Prove that f is continuous.

7. (a) Let X be a set and d the distance function on X defined by $d(x, x) = 0, d(x, y) = 1$ for $x \neq y$. Prove that each subset of (X, d) is open.

   (b) Let A be a closed, non-empty subset of the real numbers that has a lower bound. Prove that A contains its greatest lower bound.

8. (a) For $i = 1, 2, \ldots, n$ let the metric space (X_i, d_i) be topologically equivalent to the metric space (Y_i, d_i'). Prove that if $X = \prod_{i=1}^n X_i$ and $Y = \prod_{i=1}^n Y_i$ are converted into metric spaces in the standard manner, then these two metric spaces are topologically equivalent.

   (b) Let (Y, d') be a subspace of (X, d). Let $a_1, a_2, \ldots$ be a sequence of points of Y and let $a \in Y$. Prove that if $\lim_n a_n = 1$ in (Y, d') then $\lim_n a_n = a$ in (X, d).