HOMEWORK 2: SOLUTIONS - MATH 490
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Problem 1 *Let X be the set of continuous functions $f : [a, b] \to \mathbb{R}$. Let d^* be the distance function on X defined by $d^*(f, g) = \int_a^b |f(t) - g(t)|dt$, for $f, g \in X$. For each $f \in X$, set $I(f) = \int_a^b f(t)dt$. Prove that the function $I : (X, d^*) \to (\mathbb{R}, d)$ is continuous.*

Solution:

Let $f \in X$ and $\epsilon > 0$. Take $\delta = \epsilon > 0$. Then, we have, for all $g \in X$, with $d^*(f, g) < \delta$, i.e., with $\int_a^b |f(t) - g(t)|dt < \epsilon$,

$$\begin{array}{rcl} d(I(f), I(g)) & = & |\int_a^b f(t)dt - \int_a^b g(t)dt| \\ & = & |\int_a^b (f(t) - g(t))dt| \\ & \leq & \int_a^b |f(t) - g(t)|dt \\ & < & \epsilon. \end{array}$$

Therefore I is continuous at f and, since f was arbitrary, $I : (X, d^*) \to (\mathbb{R}, d)$ is continuous.

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Problem 2 *Let $(X_i, d_i), (Y_i, d'_i), i = 1, 2, \ldots, n$ be metric spaces. Let $f_i : X_i \to Y_i, i = 1, \ldots, n$ be continuous functions. Let $X = \prod_{i=1}^n X_i$ and $Y = \prod_{i=1}^n Y_i$ and convert X and Y into metric spaces in the standard manner. Define the function $F : X \to Y$ by*

$$F(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n)).$$

Prove that F is continuous.

Solution:

Let $(x_1, \ldots, x_n) \in X$ and $\epsilon > 0$. Since $f_i : X_i \to Y_i$ is continuous, for all $1 \leq i \leq n$, there exist $\delta_i > 0$, such that, for all $y_i \in X_i$, with $d_i(x_i, y_i) < \delta_i$, $d'_i(f_i(x_i), f_i(y_i)) < \epsilon$. Let $\delta = \min_i \delta_i$. Then, for all $(y_1, \ldots, y_n) \in X$, with $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) < \delta$, we have, $\max_i d_i(x_i, y_i) < \delta$, whence $d_i(x_i, y_i) < \delta < \delta_i$ and therefore $d'_i(f_i(x_i), f_i(y_i)) < \epsilon$. Hence

$$d'(F(x_1, \ldots, x_n), F(y_1, \ldots, y_n)) = \max_i d'_i(f_i(x_i), f_i(y_i)) < \epsilon,$$

which shows that F is continuous. ■

Problem 3 *Let (X, d) be a metric space such that $d(x, y) = 1$ whenever $x \neq y$. Let $a \in X$. Prove that $\{a\}$ is a neighborhood of a and constitutes a basis for the system of neighborhoods at a. Prove that every subset of X is a neighborhood of each of its points.*

Solution:

$\{a\}$ is a neighborhood of a since it contains the open ball $B(a; \frac{1}{2})$. In fact, since for all $x \neq a, d(x, a) = 1$, we have $B(a; \frac{1}{2}) = \{a\}$. Suppose that U is a neighborhood of a. Then, there exists an open ball $B(a;r)$, such that $a \in B(a;r) \subseteq U$. We have that $B(a;r) = \{a\}$, if $r < 1$ and $B(a;r) = X$, if $r \geq 1$, whence, in every case $\{a\} \subseteq B(a;r)$ and $\{a\}$ is a basis for the system of neighborhoods at a.

Let $Y \subseteq X$. Then, for all $y \in Y$, $y \in \{y\} \subseteq Y$, whence Y is a neighborhood of y and, since $y \in Y$ was arbitrary, Y is a neighborhood of each of its points. ■

Problem 4 *1. Let a be a point in a metric space X. Let N be the set of positive integers. Prove that there is a collection $\{B_n\}_{n \in N}$ of neighborhoods of a which constitutes a basis for the system of neighborhoods at a.*

2. Let a and b be distinct points of a metric space X. Prove that there are neighborhoods N_a and N_b of a and b, respectively, such that $N_a \cap N_b = \emptyset$.

Solution:

1. Consider the collection $\{B_n\}_{n \in N}$, with $B_n = B(a; \frac{1}{n})$. We show that this collection is a basis for the system of neighborhoods at a. Suppose that U is a neighborhood at a. Then there exists $\epsilon > 0$, such that $B(a; \epsilon) \subseteq U$. Take $n > \frac{1}{\epsilon}$, i.e., $\frac{1}{n} < \epsilon$. We then have $B(a; \frac{1}{n}) \subseteq B(a; \epsilon) \subseteq U$, whence U contains a member of $\{B_n\}_{n \in N}$, which, therefore, constitutes a basis for the system of neighborhoods at a.

2. Since $a \neq b$, we must have $d(a,b) > 0$. Consider the two balls $N_a = B(a, \frac{d(a,b)}{2})$ and $N_b = B(b, \frac{d(a,b)}{2})$. Then $a \in N_a, b \in N_b$, and we also have, for all $x \in X$, such that $x \in N_a \cap N_b$, $x \in N_a$ and $x \in N_b$, whence $d(a,x) < \frac{d(a,b)}{2}$ and $d(b,x) < \frac{d(a,b)}{2}$, whence $d(a,b) \leq d(a,x) + d(x,b) < \frac{d(a,b)}{2} + \frac{d(a,b)}{2} = d(a,b)$, a contradiction. Therefore $N_a \cap N_b = \emptyset$.

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Problem 5 *1. Let $X_1, X_2, \ldots, X_k$ be metric spaces and convert $X = \prod_{i=1}^k X_i$ into a metric space in the standard manner. Each of the points $a_1, a_2, \ldots$ of a sequence of points of X has k coordinates; that is $a_n = (a_1^n, \ldots, a_k^n) \in X, n = 1, 2, \ldots$. Let $c = (c_1, c_2, \ldots, c_k) \in X$. Prove that $\lim_n a_n = c$ if and only if $\lim_n a_i^n = c_i, i = 1, 2, \ldots, k$.*

2. Prove that a subsequence of a convergent sequence is convergent and converges to the same limit as the original sequence.

Solution:

1. Suppose that $\lim_n a_n = c$. Then, for all $\epsilon > 0$, there exist $N > 0$, such that, for all $n > N$, $d(a_n, c) < \epsilon$, i.e., $\max_i d_i(a_i^n, c_i) < \epsilon$. Thus $d_i(a_i^n, c_i) \leq \max_i d_i(a_i^n, c_i) < \epsilon$, which shows that $\lim_n a_i^n = c_i, i = 1, 2, \ldots, n$.

   Suppose, conversely, that $\lim_n a_i^n = c_i, i = 1, \ldots, n$. Then, for all $\epsilon > 0$, there exist $N_i > 0$, such that, for all $n > N_i$, $d_i(a_i^n, c_i) < \epsilon$, $i = 1, \ldots, n$. Take $N = \max_i N_i$. Then, for all $n > N \geq N_i$, we have $d(a_n, c) = \max_i d_i(a_i^n, c_i) < \epsilon$ and, therefore, $\lim_n a_n = c$.

2. Suppose that, for all $\epsilon > 0$, there exists $N > 0$, such that, for all $n > N$, $d(a_n, l) < \epsilon$. Consider a subsequence $\{a_{j_n}\}_{n\in\mathbf{N}}$ of $\{a_n\}_{n\in\mathbf{N}}$. Then, setting $N' : j_n > N$, for all $n > N'$, we obtain $d(a_{j_n}, l) < \epsilon$, whence $\{a_{j_n}\}$ is also convergent with the same limit as $\{a_n\}$.

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Problem 6 *1. A sequence of real numbers $a_1, a_2, \ldots$ is called* monotone non-decreasing *if $a_i \leq a_{i+1}$ for each i and called* monotone non-increasing *if $a_i \geq a_{i+1}$ for each i. A sequence which is either monotone non-decreasing or monotone non-increasing is called* monotone. *The sequence is said to be* bounded above *if there is a number K such that $a_i \leq K$ for each i and* bounded below *if there is a number M such that $a_i \geq M$ for each i. A sequence which is both bounded above and bounded below is called* bounded. *Prove that a convergent sequence of real numbers is bounded. Prove that a monotone non-decreasing sequence of real numbers which is bounded above converges to a limit a and that a is the least upper bound of the set $\{a_1, a_2, \ldots\}$.*

2. Let A be a nonempty subset of a metric space (X, d). Define the function $f : X \to \mathbb{R}$ by $f(x) = d(x, A)$. Prove that f is continuous.

Solution:

1. Suppose that $\{a_n\}_{n\in\mathbf{N}}$ is a convergent sequence of real numbers with limit l. Then, given $\epsilon > 0$, there exists $N > 0$, such that for all $n > N$, $|a_n - l| < \epsilon$, i.e., $l - \epsilon < a_n < l + \epsilon$. Now let $M = \min\{l - \epsilon, a_1, \ldots, a_N\}$ and $K = \max\{l + \epsilon, a_1, \ldots, a_N\}$. Then we have that $M \leq a_n \leq K$, for all $n \in \mathbf{N}$, whence $\{a_n\}$ is bounded.

   Suppose now that $\{a_n\}$ is non-decreasing and bounded above and let l be a least upper bound. Then $a_n \leq l$, for all $n \in \mathbf{N}$, and, since l is a least upper bound, for all $\epsilon > 0$, there exists $N > 0$, such that $a_N > l - \epsilon$. But then, for all $n > N$, we have, by the bound property and by monotonicity, $l - \epsilon < a_N \leq a_n \leq l < l + \epsilon$, i.e., $|a_n - l| < \epsilon$. Therefore $\{a_n\}$ converges to l.

2. Let $x \in X$. Since $d(y, A) = \inf_{a\in A} d(y, A)$, for all $\eta > 0$, there exists $a_\eta \in A$, such that $d(y, a_\eta) < d(y, A) + \eta$. Now let $\epsilon > 0$ and suppose that $y \in X$, such that $d(x, y) < \delta = \epsilon$ and also, without loss of generality, that $d(x, A) \geq d(y, A)$. Then

$$\begin{aligned}
|f(x) - f(y)| &= |d(x, A) - d(y, A)| \\
&= d(x, A) - d(y, A) \\
&= \inf_{a\in A} d(x, a) - d(y, A) \\
&\leq d(x, a_\eta) - d(y, A) \\
&\leq d(x, y) + d(y, a_\eta) - d(y, A) \\
&< \epsilon + \eta.
\end{aligned}$$

   Since $\eta > 0$ is arbitrary, this yields that $|f(x) - f(y)| \leq \epsilon$, and, therefore, f is continuous.

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Problem 7 *1. Let X be a set and d the distance function on X defined by $d(x,x) = 0, d(x,y) = 1$ for $x \neq y$. Prove that each subset of (X,d) is open.*

2. *Let A be a closed, non-empty subset of the real numbers that has a lower bound. Prove that A contains its greatest lower bound.*

Solution:

1. Let $A \subseteq X$. Then $A = \bigcup_{a\in A}\{a\} = \bigcup_{a\in A} B(a; \frac{1}{2})$, whence A is open as the union of open sets.

2. Let $b = \inf A$. Then, for all $n > 0$, there exists $a_n \in A$, such that $a_n < b + \frac{1}{n}$, i.e., $|a_n - b| < \frac{1}{n}$. Thus $\{a_n\}_{n\in\mathbf{N}}$ is a sequence in A that converges to b. But A is closed, whence $b \in A$.

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Problem 8 *1. For $i = 1, 2, \ldots, n$ let the metric space (X_i, d_i) be topologically equivalent to the metric space (Y_i, d'_i). Prove that if $X = \prod_{i=1}^n X_i$ and $Y = \prod_{i=1}^n Y_i$ are converted into metric spaces in the standard manner, then these two metric spaces are topologically equivalent.*

2. *Let (Y, d') be a subspace of (X, d). Let $a_1, a_2, \ldots$ be a sequence of points of Y and let $a \in Y$. Prove that if $\lim_n a_n = a$ in (Y, d') then $\lim_n a_n = a$ in (X, d).*

Solution:

1. Since X_i and Y_i are topologically equivalent, there exist bijections $f_i : X_i \to Y_i$, and a constant k_i, such that, for all $x_i, y_i \in X_i$, $d'_i(f_i(x_i), f_i(y_i)) < k_i d_i(x_i, y_i)$. Define $F : X \to Y$ by $F(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n))$, for all $(x_1, \ldots, x_n) \in X$, and set $K = \max_i k_i$. Then we have that $F : X \to Y$ is a bijection such that

$$\begin{array}{rcl} d'(F(x_1,\ldots,x_n), F(y_1,\ldots,y_n)) & = & \max_i d'_i(f_i(x_i), f_i(y_i)) \\ & < & \max_i k_i d_i(x_i, y_i) \\ & \leq & K \max_i d_i(x_i, y_i) \\ & = & Kd((x_1,\ldots,x_n),(y_1,\ldots,y_n)). \end{array}$$

   The inverse function property with the corresponding bound law may be proved very similarly.

2. Let $\epsilon > 0$. Since $\lim_n a_n = a$ in Y, there exists $N > 0$, such that, for all $n > N$, $d'(a_n, a) < \epsilon$. But then $d(a_n, a) = d'(a_n, a) < \epsilon$, whence $\lim_n a_n = a$ in X also.

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