HOMEWORK 4 - MATH 490

DUE DATE: Monday, March 17
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work.

GOOD LUCK!!

1. Prove that a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a homeomorphism if and only if
   (a) f is one-one;
   (b) f is onto;
   (c) For each point $x \in X$ and each subset N of X, N is a neighborhood of x if and only if $f(N)$ is a neighborhood of $f(x)$.
2. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a homeomorphism. Let a third topological space $(Z, \mathcal{T}'')$ and a function $h : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$ be given. Prove that h is continuous if and only if hf is continuous. Let another function $k : (Z, \mathcal{T}'') \to (X, \mathcal{T})$ be given. Prove that k is continuous if and only if fk is continuous.
3. If Y is a subspace of X and Z is a subspace of Y, then Z is a subspace of X.
4. (a) Let O be an open subset of a topological space X. Prove that a subset A of O is relatively open in O if and only if it is an open subset of X.
   (b) Let F be a closed subset of a topological space X. Prove that a subset A of F is relatively closed in F if and only if it is a closed subset of X.
5. Let Y be a subspace of X and let A be a subset of Y. Denote by $\mathrm{Int}_X(A)$ the interior of A in the topological space X and by $\mathrm{Int}_Y(A)$ the interior of A in the topological space Y. Prove that $\mathrm{Int}_X(A) \subset \mathrm{Int}_Y(A)$. Illustrate by an example the fact that in general $\mathrm{Int}_X(A) \neq \mathrm{Int}_Y(A)$.
6. Let $X = \prod_{\alpha \in I} X_\alpha$ be the topological product of the family of spaces $\{X_\alpha\}_{\alpha \in I}$. Prove that a function $f : Y \to X$ from a space Y into the

product X is continuous if and only if for each $\alpha \in I$ the function $f_\alpha = p_\alpha f : Y \to X_\alpha$ is continuous.

7. Let $\{X_\alpha\}_{\alpha \in I}$ and $\{Y_\alpha\}_{\alpha \in I}$ be two families of spaces indexed by the same indexing set I. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a continuous function. Define $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ by $(f(x))(\alpha) = f_\alpha(x(\alpha))$. Prove that f is continuous.

8. Let n be an integer. Let $\phi_n : \mathbb{R} \to \mathbb{R}$ be the function from the real line into itself defined by $\phi_n(x) = nx$. Let $p(t) = (\cos(2\pi t), \sin(2\pi t))$ as before. Show that ϕ_n induces a function $\Phi_n : S \to S$ of the circle into itself so that $\Phi_n p = p\phi_n$.