HOMEWORK 4: SOLUTIONS - MATH 490
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Problem 1 *Prove that a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a homeomorphism if and only if*

1. *f is one-one;*

2. *f is onto;*

3. *For each point $x \in X$ and each subset N of X, N is a neighborhood of x if and only if $f(N)$ is a neighborhood of $f(x)$.*

Solution:

The only if direction is easy. For the if direction we need to show that both $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous. Suppose $O \subseteq Y$ is open. Then, if $O = \emptyset$, $f^{-1}(O) = \emptyset$ is open. If $O \neq \emptyset$, then, let $x \in O$. O is then a neighborhood of x. Hence $f^{-1}(O)$ is a neighborhood of $f^{-1}(x)$, since $O = f(f^{-1}(O))$ and $x = f(f^{-1}(x))$. Since $x \in O$ is arbitrary, $f^{-1}(O)$ is a neighborhood of each of its points and therefore open. Thus f is continuous. A similar argument applies to show that $f^{-1} : Y \to X$ is also continuous. Hence f is a homeomorphism. ■

Problem 2 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a homeomorphism. Let a third topological space $(Z, \mathcal{T}'')$ and a function $h : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$ be given. Prove that h is continuous if and only if hf is continuous. Let another function $k : (Z, \mathcal{T}'') \to (X, \mathcal{T})$ be given. Prove that k is continuous if and only if fk is continuous.*

Solution:

If $h : Y \to Z$ is continuous, then hf is continuous, since the composite of two continuous functions is continuous. If, conversely, hf is continuous, then $h = (hf)f^{-1}$ is also continuous for the same reason.

The second assertion may be proved similarly. ■

Problem 3 *If Y is a subspace of X and Z is a subspace of Y, then Z is a subspace of X.*

Solution:

Let O be open in Z. Then, there exists an open P in Y, such that $O = P \cap Z$. Finally, there exists an open Q in X, such that $P = Q \cap Y$. But then $O = P \cap Z = (Q \cap Y) \cap Z = Q \cap (Y \cap Z) = Q \cap Z$. Thus O is the intersection of an open in X with Z.

Conversely, suppose that $O = Q \cap Z$ for an open set Q in X. Then $Q \cap Y$ is open in Y and $O = (Q \cap Y) \cap Z$, whence O is relatively open in Z. ■

Problem 4

1. *Let O be an open subset of a topological space X. Prove that a subset A of O is relatively open in O if and only if it is an open subset of X.*

2. *Let F be a closed subset of a topological space X. Prove that a subset A of F is relatively closed in F if and only if it is a closed subset of X.*

Solution:

Let A be relatively open in O. This is the case if and only if, there exists an open A' in X, such that $A' \cap O = A$. Thus A is open in X. Conversely, if $A \subset O$ is open in X, then $A \cap O = A$ is relatively open in O.

The second part may be proved similarly. ∎

Problem 5 *Let Y be a subspace of X and let A be a subset of Y. Denote by $\mathrm{Int}_X(A)$ the interior of A in the topological space X and by $\mathrm{Int}_Y(A)$ the interior of A in the topological space Y. Prove that $\mathrm{Int}_X(A) \subset \mathrm{Int}_Y(A)$. Illustrate by an example the fact that in general $\mathrm{Int}_X(A) \neq \mathrm{Int}_Y(A)$.*

Solution:

If $x \in \mathrm{Int}_X(A)$, then, there exists O open in X, such that $x \in O \subset A$. Thus $O \cap Y$ is open in Y, such that $x \in O \cap Y \subset A \cap Y = A$. Thus $x \in \mathrm{Int}_Y(A)$.

Note that $\mathrm{Int}_{\mathbb{R}}([0,1]) = (0,1)$, whereas $\mathrm{Int}_{[0,1]}([0,1]) = [0,1]$. ∎

Problem 6 *Let $X = \prod_{\alpha \in I} X_\alpha$ be the topological product of the family of spaces $\{X_\alpha\}_{\alpha \in I}$. Prove that a function $f : Y \to X$ from a space Y into the product X is continuous if and only if for each $\alpha \in I$ the function $f_\alpha = p_\alpha f : Y \to X_\alpha$ is continuous.*

Solution:

(This solution is taken from *Topology*, Second Edition, James R. Munkres, Prentice Hall). Each of the projections $p_\alpha, \alpha \in I$, is continuous, for, if U_α is open in X_α, then $p_\alpha^{-1}(U_\alpha)$ is a subbasic element of the product topology. Hence, if $f : Y \to X$ is continuous, $f_\alpha = p_\alpha f$ is continuous as the composite of two continuous functions, for all $\alpha \in I$.

Conversely, assume that each of the f_α's is continuous. To prove that f is continuous, it suffices to show that the inverse image under f of each subbasis element is open in Y. A typical subbasis element for the product topology is of the form $p_\alpha^{-1}(U_\alpha)$, where $\alpha \in I$ and U_α is open in X_α. Now $f^{-1}(p_\alpha^{-1}(U_\alpha)) = f_\alpha^{-1}(U_\alpha)$, since $f_\alpha = p_\alpha f$. Since f_α is assumed continuous, this set is open in Y. ∎

Problem 7 *Let $\{X_\alpha\}_{\alpha \in I}$ and $\{Y_\alpha\}_{\alpha \in I}$ be two families of spaces indexed by the same indexing set I. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a continuous function. Define $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ by $(f(x))(\alpha) = f_\alpha(x(\alpha))$. Prove that f is continuous.*

Solution:

Let $p_\alpha : X \to X_\alpha, \alpha \in I$, and $q_\alpha : Y \to Y_\alpha, \alpha \in I$, be the projection maps. Since $p_\alpha : X \to X_\alpha$ is continuous for all $\alpha \in I$, the map $f_\alpha p_\alpha : X \to Y_\alpha$ is continuous, for all $\alpha \in I$. But $f_\alpha p_\alpha = q_\alpha f$, for all $\alpha \in I$, whence $q_\alpha f : X \to Y_\alpha$ is continuous, for all $\alpha \in I$. Thus, again by the previous problem, the map $f : X \to Y$ is also continuous. ∎

Problem 8 *Let n be an integer. Let $\phi_n : \mathbb{R} \to \mathbb{R}$ be the function from the real line into itself defined by $\phi_n(x) = nx$. Let $p(t) = (\cos(2\pi t), \sin(2\pi t))$ as before. Show that ϕ_n induces a function $\Phi_n : S \to S$ of the circle into itself so that $\Phi_n p = p\phi_n$.*

Solution:

Define $\Phi_n : S \to S$, by $\Phi_n(\cos\phi, \sin\phi) = (\cos n\phi, \sin n\phi)$, for all $\phi \in [0, 2\pi)$. We then have

$$\begin{aligned} \Phi_n(p(t)) &= \Phi_n(\cos(2\pi t), \sin(2\pi t)) \\ &= (\cos(2\pi nt), \sin(2\pi nt)) \\ &= p(nt) \\ &= p(\phi_n(t)). \end{aligned}$$

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