HOMEWORK 5: SOLUTIONS - MATH 490
INSTRUCTOR: George Voutsadakis

Problem 1 *On the real line, prove that the set of non-zero numbers is not a connected set.*

Solution:
The space $(-\infty, 0) \cup (0, \infty)$ is disconnected, since $-\infty, 0)$ and $(0, \infty)$ are nonempty disjoint relatively open in $(-\infty, 0) \cup (0, \infty)$ and their union is the whole space. ■

Problem 2 *Let A and B be subsets of a topological space X. if A is connected, B is open and closed, and $A \cap B \neq \emptyset$, prove that $A \subset B$. (Hint: Assume $A \not\subset B$ and use the sets $P = A \cap B$ and $Q = A \cap C(B)$ to prove that A is not connected.)*

Solution:
Suppose $A \not\subset B$. Consider $P = A \cap B$ and $Q = A \cap C(B)$. Then, by the hypothesis, $P \neq \emptyset$ and, by our assumption, $Q \neq \emptyset$. B being both open and closed in X yields that both P and Q are relatively open in A and, obviously, $P \cup Q = A, P \cap Q = \emptyset$. Therefore A is disconnected, a contradiction. Hence $A \subset B$. ■

Problem 3 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that the image under f of each interval is either a single point or an interval.*

Solution:
Each interval I is connected. Therefore, since f is continuous, $f(I)$ is also connected. Hence, since $f(I) \neq \emptyset$, $f(I)$ is either a single point or an interval. ■

Problem 4 *Prove that a homeomorphism $f : [a, b] \to [a, b]$ carries end points into end points.*

Solution:
Suppose that $f(a) = c$, with $a < c < b$. Then, there exists some $\epsilon > 0$, such that $(c - \epsilon, c + \epsilon) \subset [a, b]$. Thus, since f is a homeomorphism, there exists $\delta > 0$ and $\eta > 0$, such that $f((a, a + \delta)) \subset (c - \epsilon, c)$ and $f((a, a + \eta)) \subset (c, c + \epsilon)$. But then, for all $x \in (a, a + \min(\delta, \eta))$, we have $c < f(x) < c$ a contradiction. ■

Problem 5 *Prove that a polynomial of odd degree considered as a function from the reals to the reals has at least one real root.*

Solution:
Let $f(x)$ be a polynomial of odd degree. Then $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$ (or $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$, depending on whether the leading coefficient is positive or negative, respectively). Hence, there exist $a, b \in \mathbb{R}$, such that $f(a) < 0$ and $f(b) > 0$. Now the Intermediate Value Theorem applies to give an $x \in (a, b)$, such that $f(x) = 0$. ■

Problem 6 *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function from a closed interval into the reals. Let* $U = f(u)$ *and* $V = f(v)$ *be such that* $U \leq f(x) \leq V$ *for all* $x \in [a, b]$*. Prove that there is a* w *between* u *and* v *such that* $f(w) \cdot (b - a) = \int_a^b f(t)dt$*.*

Solution:

Since $U \leq f(t) \leq V$, for all $t \in [a, b]$, we have that $\int_a^b U dt \leq \int_a^b f(t)dt \leq \int_a^b V dt$, i.e., that $U(b-a) \leq \int_a^b f(t)dt \leq V(b-a)$. Therefore $U \leq \frac{\int_a^b f(t)dt}{b-a} \leq V$. Thus, by the intermediate value theorem, there exists $w \in [u, v]$, such that $f(w) = \frac{\int_a^b f(t)dt}{b-a}$, i.e., $f(w)(b - a) = \int_a^b f(t)dt$. ■

Problem 7 *Prove that a nonempty connected subset of a topological space that is both open and closed is a component.*

Solution:

Suppose that X is both open and closed but that it is not a component. Since X is connected, it is properly contained in a connected component D. But then $P = X \cap D$ and $Q = C(X) \cap D$ are two nonempty open disjoint subsets of D with $P \cup Q = D$. Thus D is disconnected, a contradiction. ■

Problem 8 *Let* X *be a topological space that has a finite number of components. Prove that each component of* X *is both open and closed.*

Solution:

We know that each connected component is closed. But in case the connected components are finitely many, each connected component is the complement of the union of finitely many closed sets, which is closed. Therefore, it is also open. ■