HOMEWORK 6 - MATH 490

DUE DATE: Monday, April 14
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work.

GOOD LUCK!!

1. Prove that the product of two locally connected topological spaces is locally connected.

2. Verify that in a topological space X
   (a) if there is a path with initial point A and terminal point B, then there is a path with initial point B and terminal point A, and
   (b) if there is a path connecting points A and B and a path connecting points B and C, then there is a path connecting points A and C.

3. If A and B are path-connected subsets of a topological space X and $A \cap B \neq \emptyset$, then $A \cup B$ is path-connected.

4. Let X, Y be topological spaces and $f : X \to Y$ be a continuous function with $f(x) = y$. Let g, g' be closed paths at $x \in X$. Prove that $fg \cong fg'$ whenever $g \cong g'$.

5. Two groups G and G' are called *isomorphic* if there are homomorphisms $h : G \to G'$ and $h' : G' \to G$ such that $h'h$ is the identity mapping on G and hh' is the identity mapping on G'. Prove that if $f : X \to Y$ is a homeomorphism of the topological space X with the space Y such that $f(x) = y$, then $\Pi(X, x)$ is isomorphic to $\Pi(Y, y)$.

6. An isomorphism of a group G with itself is called an *automorphism*. Let f and f' be paths in a space Z with $f(0) = f'(1) = z$ and $f(1) = f'(0) = y$. Let $f' \cdot f^{-1}$ be the path defined by

$$(f' \cdot f^{-1})(t) = \begin{cases} f'(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ f^{-1}(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

   Prove that $a_{f'}a_f$ is an automorphism of $\Pi(Z, y)$ such that $a_{f'}a_f([g]) = [f' \cdot f^{-1}] \cdot [g] \cdot [f' \cdot f^{-1}]^{-1}$.

7. (a) Prove that the real line $\mathbb{R}$ is not compact.
   (b) Prove that every finite subset of a topological space is compact.

8. Let X be a topological space. A family $\{F_\alpha\}_{\alpha \in I}$ of subsets of X is said to have the *finite intersection property* if for each finite subset J of I, $\cap_{\alpha \in J} F_\alpha \neq \emptyset$. Prove that X is compact if and only if for each family $\{F_\alpha\}_{\alpha \in I}$ of closed subsets of X that has the finite intersection property, we have $\cap_{\alpha \in I} F_\alpha \neq \emptyset$.