HOMEWORK 7 - MATH 490

DUE DATE: Monday, April 28
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. Each question is worth 5 points. It is necessary to show your work.

GOOD LUCK!!

1. Let X be a compact space and $(F_n)_{n=1,2,3,\ldots}$ a sequence of nonempty closed subsets of X such that $F_{n+1} \subset F_n$ for each n. Prove that $\cap_{n=1}^{\infty} F_n \neq \emptyset$.

2. Let $f : [a,b] \to \mathbb{R}$ be continuous. Prove that the set $f([a,b])$ has both a least upper bound M and a greatest lower bound m and that there are points $u, v \in [a,b]$ such that $f(u) = M, f(v) = m$.

3. A topological space X is said to be *locally compact* if each point $x \in X$ has at least one compact neighborhood. Prove that the real line and $\mathbb{R}^n$ are locally compact.

4. In a metric space (X, d) a sequence $a_1, a_2, \ldots$ of points of X is called a *Cauchy sequence* if for each $\epsilon > 0$ there is a positive integer N such that $d(a_n, a_m) < \epsilon$ whenever $m, n > N$. A metric space X is called *complete* if every Cauchy sequence in X converges to a point of X. Prove that a compact metric space is complete.

5. Let (X, d) be a compact metric space. Prove that X is "bounded with respect to d"; that is, there is a positive number K such that $d(x, y) \leq K$ for all $x, y \in X$.

6. Let X be an arbitrary non-empty set and $f : X \to 2^X$ an arbitrary function from X to the subsets of X. Let A be the subset of X consisting of those points $x \in X$ such that $x \notin f(x)$. Prove that there cannot be a point $a \in X$ such that $A = f(a)$. Finally, prove that there is no onto function $f : X \to 2^X$.

7. Let a function $f : N \to [0,1]$ be given, N the set of positive integers. In the resulting enumeration $x_1 = f(1), x_2 = f(2), \ldots,$ of numbers in $[0,1]$, express each number x_k in decimal notation $x_k =$

$0.a_1^k a_2^k \ldots a_n^k \ldots$, a_n^k an integer $0 \leq a_n^k \leq 9$. Construct a real number $y = 0.y_1 y_2 \ldots y_n \ldots$ such that $y_r \neq a_r^r, r = 1, 2, \ldots$, thereby obtaining the result that f cannot be onto and consequently the real numbers are not countable.

8. Let X and Y be topological spaces satisfying the second axiom of countability. Prove that $X \times Y$ also satisfies the second axiom of countability and hence $\mathbb{R}^n$ does.