HOMEWORK 3 - MATH 216

DUE DATE: After Chapter 6 is covered.
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. A few randomly selected problems will be graded for a total of 10 points. It is necessary to show your work.

GOOD LUCK!!

1. Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$
   (a) Find a_0, a_1, a_2, a_3 and a_4.
   (b) Show that $a_2 = 5a_1 - 6a_0, a_3 = 5a_2 - 6a_1$ and $a_4 = 5a_3 - 6a_2$.
   (c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

2. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if
   (a) $a_n = -n + 2$
   (b) $a_n = 5(-1)^n - n + 2$
   (c) $a_n = 3(-1)^n + 2^n - n + 2$
   (d) $a_n = 7 \cdot 2^n - n + 2$

3. Find the solution to each of the recurrence relations and initial conditions. Use an iterative approach.
   (a) $a_n = 3a_{n-1}, a_0 = 2$
   (b) $a_n = a_{n-1} + 2, a_0 = 3$
   (c) $a_n = a_{n-1} + n, a_0 = 1$
   (d) $a_n = a_{n-1} + 2n + 3, a_0 = 4$
   (e) $a_n = 2a_{n-1} - 1, a_0 = 1$
   (f) $a_n = 3a_{n-1} + 1, a_0 = 1$
   (g) $a_n = na_{n-1}, a_0 = 5$
   (h) $a_n = 2na_{n-1}, a_0 = 1$

4. A vending machine dispensing books of stamps accepts only dollar coins, $1 bills and $5 bills.
   (a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.
   (b) What are the initial conditions?
   (c) How many ways are there to deposit $10 for a book of stamps?

5. (a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0's

(b) What are the initial conditions?

(c) How many bit strings of length seven contain two consecutive 0's?

6. Solve the recurrence relations together with initial conditions given.

   (a) $a_n = a_{n-1}$ for $n \geq 1, a_0 = 2$

   (b) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2, a_0 = 1, a_1 = 0$

   (c) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 0, a_1 = 1$

   (d) $a_n = \frac{a_{n-2}}{4}$ for $n \geq 2, a_0 = 1, a_1 = 0$

7. In how many ways can a $2 \times n$ rectangular board be tiled using 1×2 and 2×2 pieces?

8. The Lucas numbers satisfy the recurrence relation $L_n = L_{n-1} + L_{n-2}$, and the initial conditions $L_0 = 2$ and $L_1 = 1$.

   (a) Show that $L_n = f_{n-1} + f_{n+1}$, for $n = 2, 3, \ldots$ where f_n is the n-th Fibonacci number.

   (b) Find an explicit formula for the Lucas numbers.

9. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9, a_1 = 10, a_2 = 32$.

10. (a) Determine the values of the constants A and B so that $a_n = An + B$ is a solution of the recurrence relation $a_n = 2a_{n-1} + n + 5$.

    (b) Use Theorem 5 on page 420 to find all the solutions of this recurrence relation.

    (c) Find the solution of this recurrence relation with $a_0 = 4$.

11. Find a closed form for the generating function for each of these sequences.

    (a) $0, 0, 0, 1, 1, 1, 1, 1, \ldots$

    (b) $0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$

    (c) $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \ldots, \binom{7}{7}, 0, 0, \ldots$

    (d) $1, 1, 0, 1, 1, 1, 1, 1, \ldots$

12. Find a closed form for the generating function of the sequence $\{a_n\}$, where

    (a) $a_n = 3^n$ for all $n = 0, 1, 2, \ldots$

    (b) $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$

    (c) $a_n = \binom{8}{n}$ for all $n = 0, 1, 2, \ldots$

    (d) $a_n = n + 4 choose n$ for all $n = 0, 1, 2, \ldots$

13. For each of these generating functions, provide a closed formula for the sequence it determines.

    (a) $(x^3 + 1)^3$

    (b) $\frac{x^3}{1+3x}$

    (c) $\frac{x^4}{1-x^4} - x^3 - x^2 - x - 1$

    (d) $2e^{2x}$

14. Find the coefficient of x^{10} in the power series of each of these functions.

    (a) $\frac{1}{1-2x}$

    (b) $\frac{1}{(1+x)^2}$

    (c) $\frac{1}{(1-x)^3}$

    (d) $\frac{1}{(1+2x)^4}$

    (e) $\frac{x^4}{(1-3x)^3}$

15. Give a combinatorial interpretations of the coefficient of x^4 in the expansion $(1 + x + x^2 + x^3 + \ldots)^3$. Use this interpretation to find this number.

16. Use generating functions to find the number of ways to make change for \$1 using

    (a) dimes and quarters

    (b) nickels, dimes and quarters

    (c) pennies, dimes and quarters

    (d) pennies, nickels, dimes and quarters.