Fourier Transforms

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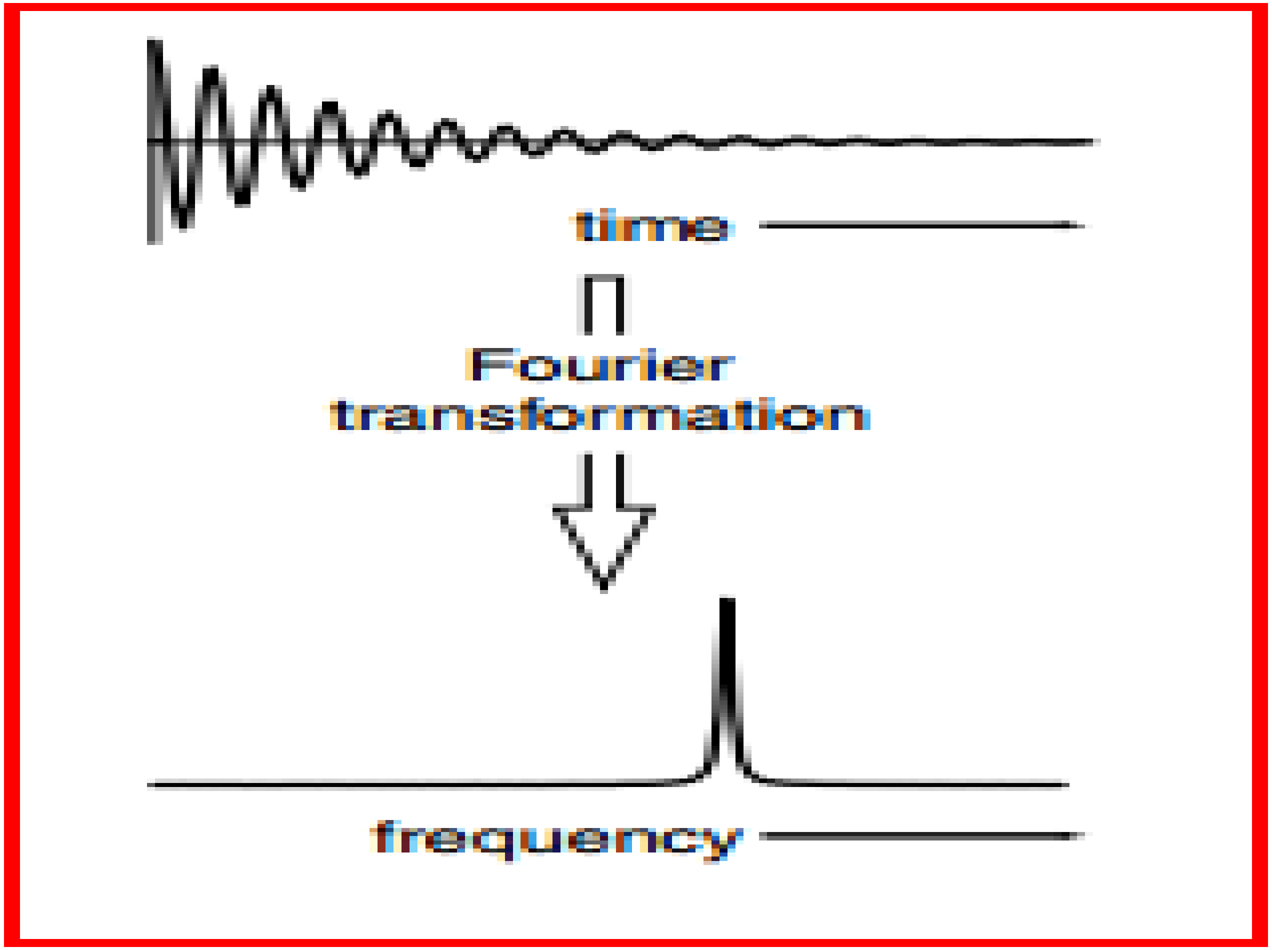
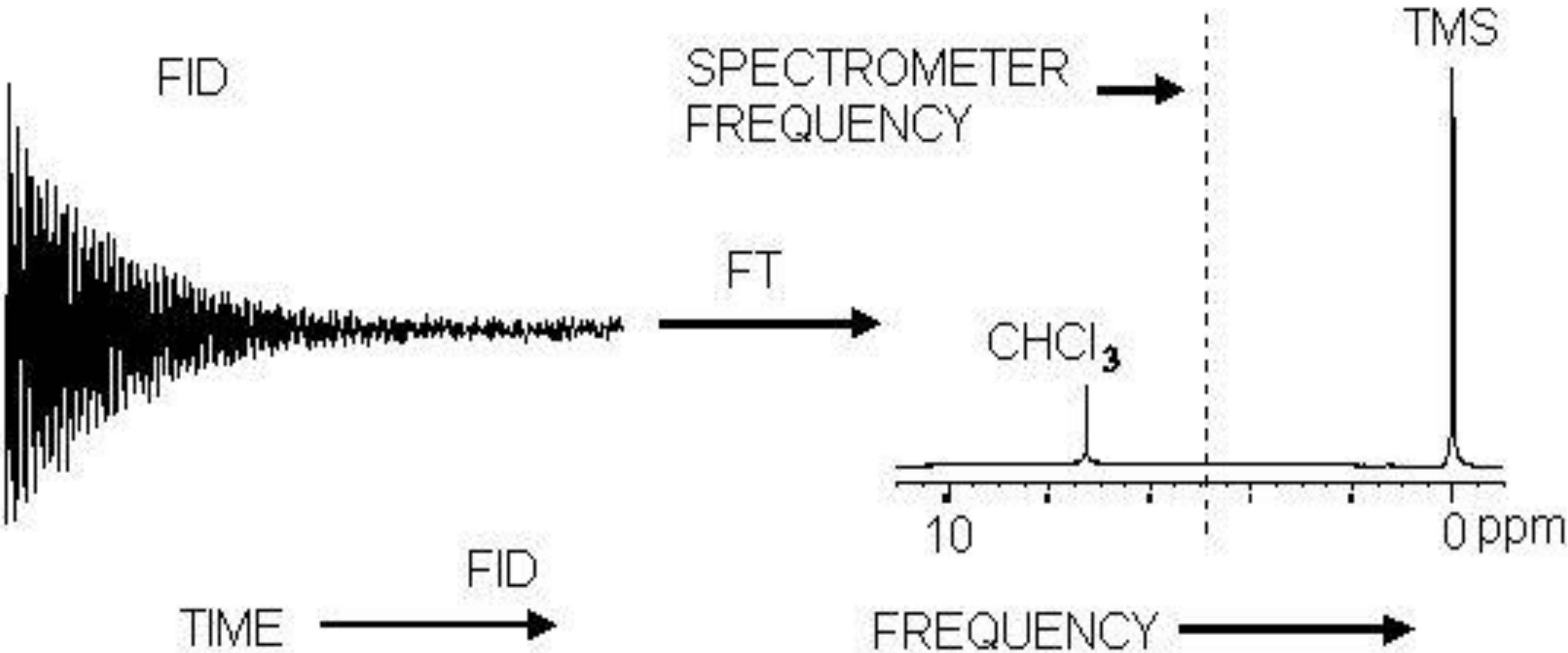
Subject of Interest



http://www-keeler.ch.cam.ac.uk/lectures/Irvine/chapter4.pdf



http://ishigirl.tripod.com/pchem/nmr.html

What is special about the Fourier Transform?

- The Fourier Transforms a time function $f(t)$ into a frequency function $F(\omega)$
- This transform is used in a variety of fields and has many applications.

Important Identity

The definition of the δ(t) function.

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0)$$

$$\phi(t) = \int_{-\infty}^{\infty} \phi(x)\delta(t-x)\,dx$$

The Fourier Integral and the Inversion Formula

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$$

Important Identity

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0)}$$

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t} dt$$

$$= e^{-i\omega(0)}$$

$$= e^{0}$$

$$F(\omega) = 1$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega$$

Proof of the Inversion Formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\right]e^{i\omega t}d\omega$$

$$\boxed{\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)dx\int_{-\infty}^{\infty} e^{i\omega(t-x)}d\omega$$

$$= \int_{-\infty}^{\infty} f(x)\delta(t-x)dx$$

$$\boxed{\phi(t) = \int_{-\infty}^{\infty}\phi(x)\delta(t-x)dx}$$

Simple Theorem

- Many Fourier Transforms are used to solve complex equations. To help to simplify the solutions of these equations, there are several simple theorems used to help in these solutions.
- All theorems are presented in the form

  $g(t) \leftrightarrow G(\omega)$ which states that $g(t)$ transforms to $G(\omega)$

Linearity Theorem

$$a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$$

- Proof

$$G(\omega) = \int_{-\infty}^{\infty} (a_1 f_1(t) + a_2 f_2(t))e^{-i\omega t} dt$$

$$G(\omega) = \int_{-\infty}^{\infty} a_1 f_1(t) e^{-i\omega t} dt + \int_{-\infty}^{\infty} a_2 f_2(t) e^{-i\omega t} dt$$

$$G(\omega) = a_1 \int_{-\infty}^{\infty} f_1(t) e^{-i\omega t} dt + a_2 \int_{-\infty}^{\infty} f_2(t) e^{-i\omega t} dt$$

$$G(\omega) = a_1 F_1(\omega) + a_2 F_2(\omega)$$

Symmetry Theorem

- If f(t) ↔ F(ω), then F(t) ↔ 2πf(-ω).

Proof

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi f(-\omega)e^{i\omega t}\,d\omega$$

$$= \int_{-\infty}^{\infty} f(-\omega)e^{i\omega t}\,d\omega$$

$$= -\int_{\infty}^{-\infty} f(u)e^{-iut}\,du$$

$$= \int_{-\infty}^{\infty} f(u)e^{-iut}\,du$$

$$g(t) = F(t)$$

Time Scaling Theorem

$f(at) \leftrightarrow \frac{1}{|a|} F(\frac{\omega}{a})$ for real values of a

Proof

$$G(\omega) = \int_{-\infty}^{\infty} f(at) e^{-i\omega t} dt$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(x) e^{-i(\omega/a)x} dx$$

$$G(\omega) = \frac{1}{a} F(\frac{\omega}{a})$$

Time Shifting Theorem

$$f(t - t_0) \leftrightarrow F(\omega)e^{-it_0\omega}$$

Proof

$$G(\omega) = \int_{-\infty}^{\infty} f(t - t_0)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(x)e^{-i\omega(t_0+x)}dx$$

$$= e^{-i\omega t_0}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$G(\omega) = F(\omega)e^{-i\omega t_0}$$

Frequency Shifting Theorem

$$e^{i\omega t_0} f(t) \leftrightarrow F(\omega - \omega_0)$$

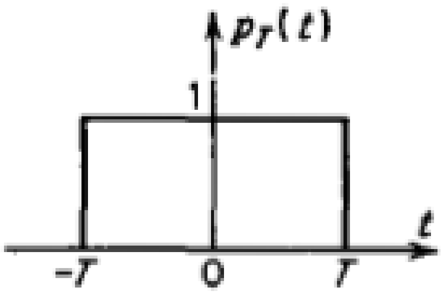
$$G(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega_0 t} e^{-i\omega t} dt$$

$$G(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i(\omega - \omega_0)t} dt$$

$$G(\omega) = F(\omega - \omega_0)$$

Example 1

$$p_T(t-t_0) \leftrightarrow \frac{2\sin\omega T}{\omega} e^{-i\omega t_0}$$



$$f(t) = p_T(t)$$

$$\int_{-\infty}^{\infty} p_T(t) e^{-i\omega t} dt = \int_{-T}^{T} e^{-i\omega t} dt$$

$$\boxed{e^{iy} = \cos y + i \sin y}$$

$$= \int_{-T}^{T} (cos\omega t - isin\omega t) dt$$

Example 1 (cont.)

$$= \int_{-T}^{T} cos\omega t dt$$

$$= \frac{1}{\omega} sin\omega t|_{-T}^{T} \qquad = \frac{2\sin\omega T}{\omega}$$

$$\boxed{f(t - t_0) \leftrightarrow F(\omega)e^{-it_0\omega}} \qquad p_T(t - t_0) \leftrightarrow \frac{2\sin\omega T}{\omega} e^{-i\omega t_0}$$

Time Differentiation Theorem

$$\frac{d^n f}{dt^n} \leftrightarrow (i\omega)^n F(\omega)$$

Proof

$$n = 0 \;\; f(t) \leftrightarrow F(\omega)$$

$$n = 1 \;\; f'(t) \leftrightarrow (i\omega)F(\omega)$$

Proof of n=1 $$\int_{-\infty}^{\infty} f'(t)e^{-i\omega t}dt$$

$$f(t)e^{-i\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}dt$$

Time Differentiation Theorem (cont.)

$$i\omega \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

Show true for n + 1

$$G(\omega) = \int_{-\infty}^{\infty} \frac{d^{n+1} f}{dt^{n+1}} e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} \frac{d}{dt}\left[\frac{d^n f}{dt^n}\right] e^{-i\omega t} dt$$

$$= \frac{d^n f}{dt^n} e^{-i\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega) \frac{d^n f}{dt^n} e^{-i\omega t}\, dt$$

Time Differentiation Theorem (cont.)

$$= (i\omega)(i\omega)^n F(\omega)$$

$$= (i\omega)^{n+1} F(\omega)$$

Frequency Differentiation Theorem

$$(-it)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}$$

Proof

$$\boxed{f(t) \leftrightarrow F(\omega) \qquad F(t) \leftrightarrow 2\pi f(-\omega)}$$

$$\frac{d^n F(t)}{dt^n} \leftrightarrow 2\pi(-it)^n f(-\omega)$$

$$G(t) = \int_{-\infty}^{\infty} (-it)^n f(-\omega) e^{i\omega t} d\omega$$

Frequency Differentiation Theorem (cont.)

$$= \int_{-\infty}^{\infty} (-it)^n f(u) e^{-iut} du$$

$$\boxed{(-it)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}}$$

$$= \frac{d^n F(t)}{dt^n}$$

Example 2

$$\frac{d^n\delta(t)}{dt^n} \leftrightarrow (i\omega)^n \qquad t^n \leftrightarrow 2\pi i^n \frac{d^n\delta(\omega)}{d\omega^n}$$

Proof

$$(it)^n \leftrightarrow 2\pi \frac{d^n\delta(-\omega)}{d(-\omega)^n}$$

$$\boxed{f(t) \leftrightarrow F(\omega) \qquad F(t) \leftrightarrow 2\pi f(-\omega)}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi \frac{d^n\delta(-\omega)}{d(-\omega)^n} e^{-i\omega t} d\omega$$

$$\boxed{(-it)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}}$$

$$\int_{-\infty}^{\infty} (-1)^n \frac{d^n}{d\omega^n} e^{-i\omega t} d\omega$$

$$(it)^n$$

Example 2 (cont.)

$$\boxed{f(t) \leftrightarrow F(\omega) \qquad F(t) \leftrightarrow 2\pi f(-\omega)}$$

$$(it)^n \leftrightarrow 2\pi \frac{d^n \delta(-\omega)}{d(-\omega)^n}$$

$$i^{2n} t^n \leftrightarrow 2\pi i^n \frac{d^n \delta(\omega)}{d(-\omega)^n}$$

$$(-1)^n t^n \leftrightarrow 2\pi i^n (-1)^n \frac{d^n \delta(\omega)}{d\omega^n}$$

$$t^n \leftrightarrow 2\pi i^n \frac{d^n \delta(\omega)}{d\omega^n}$$

Example 3

$$e^{-i\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

Proof

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{i\omega t} d\omega$$

$$= \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{i\omega t} d\omega$$

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)}$$

$$= \int_{-\infty}^{\infty} \delta(u) e^{i(u+\omega_0)t} du$$

$$= e^{i\omega_0 t}$$

Example 4

$$\frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t}) \leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

Proof

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]e^{i\omega t}d\omega$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} [\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]e^{i\omega t}d\omega$$

$$\boxed{\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{i\omega t}d\omega = e^{i\omega_0 t}}$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{i\omega t}d\omega + \int_{-\infty}^{\infty} \delta(\omega+\omega_0)e^{i\omega t}d\omega\right]$$

$$= \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})$$