HOMEWORK 1 SOLUTIONS - MATH 300
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Problem 1 *Is* $2 \in \{1,2,3\}$*? Why?*

Solution: The set $\{1,2,3\}$ has as members the three natural numbers $1, 2$ and 3. Since 2 is one of them, $2 \in \{1,2,3\}$. ■

Problem 2 *Is* $\{1,2\} \in \{\{1,2,3\},\{1,3\},1,2\}$*? Why?*

Solution: The set $\{\{1,2,3\},\{1,3\},1,2\}$ consists of four elements: the natural numbers 1 and 2 and the two sets $\{1,2,3\}$ and $\{1,3\}$. Thus, the set $\{1,2\}$ is **not** an element of the set $\{\{1,2,3\},\{1,3\}, 1,2\}$, i.e., it is not true that $\{1,2\} \in \{\{1,2,3\},\{1,3\},1,2\}$. ■

Problem 3 *Give precise descriptions in plain English of the following sets:*

(a) $\{x \in \mathbb{N} : x \textit{ is divisible by } 2 \textit{ and } x \textit{ is divisible by } 3\}$

(b) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(c) $\{(x,y) \in \mathbb{R}^2 : y = 2x \textit{ and } y = 3x\}$

Solution:

(a) Let us denote

$$\begin{aligned} A &= \{x \in \mathbb{N} : x \text{ is divisible by } 2 \text{ and } x \text{ is divisible by } 3\}; \\ B &= \{x \in \mathbb{N} : x \text{ is divisible by } 6\}. \end{aligned}$$

We show that $A = B$.

First, note that, if $x \in A$, then, there exist $m, n \in \mathbb{N}$, such that $x = 2m$ and $x = 3n$. Thus $2m = 3n$, which shows that m is divisible by 3, i.e., there exists $k \in \mathbb{N}$, such that $m = 3k$. Hence, $x = 2m = 2 \cdot 3k = 6k$. Therefore x is in fact divisible by 6, i.e., $x \in B$. This shows $A \subseteq B$.

Conversely, if $x \in B$, then, there exists $n \in \mathbb{N}$, such that $x = 6n = 2 \cdot 3 \cdot n$ and, therefore x is divisible by both 2 and 3. Thus, $x \in A$. This shows that $B \subseteq A$.

The two previous inclusions show that $A = B$.

(b) This is the set of all pairs of Cartesian coordinates of points on the plane that are on the unit circle with center at the origin.

(c) Let us denote

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 : y = 2x \text{ and } y = 3x\}; \\ B &= \{(0,0)\}. \end{aligned}$$

We show that $A = B$.

Clearly, if $(x,y) \in B$, then $x = y = 0$, whence $y = 2x$ and $y = 3x$ and, therefore $(x,y) \in A$. Thus $B \subseteq A$.

Conversely, if $(x,y) \in A$, then $y = 2x$ and $y = 3x$, whence $2x = 3x$, i.e., $x = 0$ and, hence $y = 2x = 0$. Thus $(x,y) = (0,0) \in B$. This shows that $A \subseteq B$.

The two inclusions, taken together, imply $A = B$.

■

Problem 4 *Show formally the following statements:*

(a) $\{k \in \mathbb{Z} : k = 6m \text{ for some } m \in \mathbb{Z}\} \subseteq \{k \in \mathbb{Z} : k = 2n \text{ for some } n \in \mathbb{Z}\}$;

(b) *If* $A \subsetneqq B$ *and* $B \subseteq C$, *then* $A \subsetneqq C$.

Solution:

(a) If $k \in \{k \in \mathbb{Z} : k = 6m \text{ for some } m \in \mathbb{Z}\}$, then, there exists $m \in \mathbb{Z}$, such that $k = 6m = 2(3m)$, with $3m \in \mathbb{Z}$. Thus $k \in \{k \in \mathbb{Z} : k = 2n \text{ for some } n \in \mathbb{Z}\}$. This proves the statement.

(b) We first show that $A \subseteq C$. Suppose that $x \in A$. Since $A \subsetneqq B$, we get that $x \in B$. Thus, since $B \subseteq C$, we get $x \in C$. This proves that $A \subseteq C$.

We finally show that $A \neq C$. Since $A \subsetneqq B$, there exists $x \in B$, such that $x \notin A$. But, then, since $B \subseteq C$, we get that $x \in C$ and $x \notin A$. Therefore, $A \neq C$.

■

Problem 5 *Is (each of) the following statement true for all sets A, B and C? If it is, give a proof. If it is not, provide a counterexample.*

(a) *If* $A \neq B$ *and* $B \neq C$, *then* $A \neq C$;

(b) *If* $A \in B$ *and* $B \not\subseteq C$, *then* $A \notin C$;

(c) *If* $A \subsetneqq B$ *and* $B \subseteq C$, *then* $C \not\subseteq A$;

Solution:

(a) This statement is not true for all sets A, B and C. As a counterexample, consider $A = \emptyset, B = \{0\}$ and $C = \emptyset$. Then, clearly, $A \neq B$ and $B \neq C$, but $A = C$.

(b) This statement is not true for all sets A, B and C. As a counterexample, consider $A = \emptyset$, $B = \{\emptyset, 0\}$ and $C = \{\emptyset\}$. Then, clearly, $A \in B$ and $B \not\subseteq C$, but $A \in C$.

(c) This statement is true for all sets A, B and C. Assume that $A \subsetneqq B$ and $B \subseteq C$. Since $A \subsetneqq B$, there exists $x \in B$, with $x \notin A$. Since $B \subseteq C$, $x \in C$ and $x \notin A$. Therefore $C \not\subseteq A$.

■

Problem 6 *Show that, for a set A in a universe U, we have $(A')' = A$.*

Solution: For all $x \in U$, we have

$$\begin{aligned} x \in (A')' \quad &\text{iff} \quad x \notin A' \\ &\text{iff} \quad x \in A. \end{aligned}$$

Therefore $(A')'$ and A contain exactly the same elements, i.e., $(A')' = A$. ■

Problem 7 *Show that, for any sets A, B in a universe U, we have $(A \cup B)' = A' \cap B'$.*

Solution: If $x \in (A \cup B)'$, then $x \notin A \cup B$, whence $x \notin A$ and $x \notin B$, showing that $x \in A'$ and $x \in B'$, i.e., $x \in A' \cap B'$. This proves that $(A \cup B)' \subseteq A' \cap B'$.

Conversely, if $x \in A' \cap B'$, then $x \in A'$ and $x \in B'$, whence $x \notin A$ and $x \notin B$, which gives that $x \notin A \cup B$, i.e., that $x \in (A \cup B)'$. This shows that $A' \cap B' \subseteq (A \cup B)'$.

These two parts taken together yield that $(A \cup B)' = A' \cap B'$. ■

Problem 8 *Either prove or give a counterexample for the following statement: For all sets A, B, C in a universe U, $(A\backslash B)\backslash C = A\backslash(B \cup C)$.*

Solution: Suppose that $x \in (A\backslash B)\backslash C$. Then $x \in A\backslash B$ and $x \notin C$. Therefore $x \in A$ and $x \notin B$ and $x \notin C$. The first and last two statements give, respectively, $x \in A$ and $x \notin B \cup C$. Thus, we obtain $x \in A\backslash(B \cup C)$. This proves that $(A\backslash B)\backslash C \subseteq A\backslash(B \cup C)$.

Suppose, conversely, that $x \in A\backslash(B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Therefore, $x \in A$ and $x \notin B$ and $x \notin C$. The first two statements give $x \in A\backslash B$ and the third $x \notin C$. Therefore, we obtain $x \in (A\backslash B)\backslash C$. This proves that $A\backslash(B \cup C) \subseteq (A\backslash B)\backslash C$.

The previous two statements taken together imply that $(A\backslash B)\backslash C = A\backslash(B \cup C)$. ■

Problem 9 *Consider the following three syllogisms:*

(a)	*All S is M*	(b)	*Some M is not P*	(c)	*All M is P*
	No M is P		*No M is S*		*Some S is M*
	$\therefore$ *Some S is P*		$\therefore$ *No S is P*		$\therefore$ *Some S is not P*

For each of (a),(b) *and* (c) *provide its mood, its figure and explain whether it is a valid syllogism under the modern convention regarding the empty class.*

Solution:

(a) The mood of this argument is EAI and it is of Figure 1. It is not a valid argument and the interpretation $S = M = P = \emptyset$ provides a counterexample.

(b) The mood of this argument is OEE and it is of Figure 3. It is not a valid argument and the interpretation $S = P = \{0\}$ and $M = \{1\}$ provides a counterexample.

(c) The mood of this argument is AIO and it is of Figure 1. It is not a valid argument and the interpretation $S = M = P = \{0\}$ provides a counterexample.

■

Problem 10 *Consider the following arguments*

ARGUMENT 1	ARGUMENT 2
$(A \cup C')' = 0$	$(A' \cup C' \cup D)' = 0$
$(A'C)'(BC)' = 0$	$AD = 0$
$\therefore\ (BC')' = 0$	$BC' = 0$
	$\therefore\ AB = 0$

(a) *Use a Venn diagram to determine if each argument is correct.*

(b) *If the argument is correct, then use both Boole's equational reasoning and Carroll's tree method to prove its correctness.*

Solution:

ARG. 1 This is not a valid argument. The following Venn diagram of Figure 1 provides a counterexample (i.e., take $U = A = B = C = \{0\}$):

ARG. 2 The given argument has the equivalent form:

$$\begin{aligned} &ACD' = 0 \\ &AD = 0 \\ &BC' = 0 \\ &\therefore\ AB = 0 \end{aligned}$$

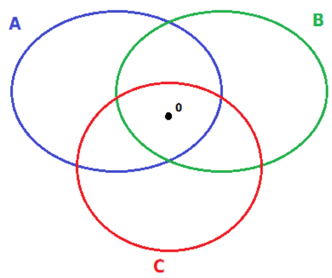

Figure 1: Venn Diagram for ARGUMENT 1

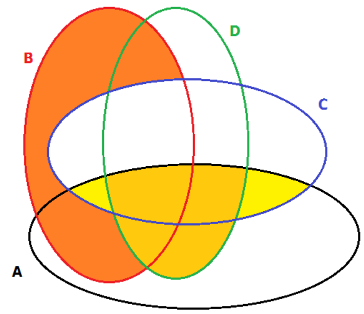

Figure 2: Venn Diagram for ARGUMENT 2

This is a valid argument as shown by the Venn diagram of Figure 2. The Boolean equational reasoning that yields the conclusion from the premises is

$$
\begin{array}{rcll}
AB & = & AB1 & \text{(Intersection with Universe)} \\
 & = & AB(C \cup C') & \text{(Union of } C \text{ and } C'\text{)} \\
 & = & ABC \cup ABC' & \text{(Distributivity)} \\
 & = & ABC1 \cup ABC' & \text{(Intersection with Universe)} \\
 & = & ABC(D \cup D') \cup ABC' & \text{(Union of } D \text{ and } D'\text{)} \\
 & = & ABCD \cup ABCD' \cup ABC' & \text{(Distrubutivity)} \\
 & = & BC0 \cup B0 \cup A0 & \text{(Premises)} \\
 & = & 0 \cup 0 \cup 0 & \text{(Intersection with } \emptyset\text{)} \\
 & = & 0 & \text{(Union).}
\end{array}
$$

Finally, Carroll's tree yielding the same conclusion is given in Figure 3

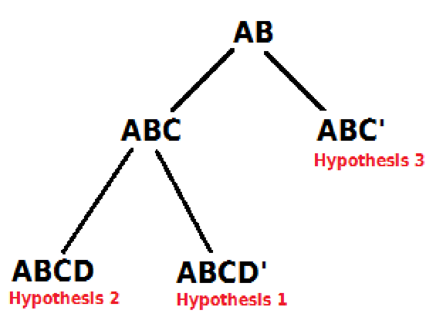

Figure 3: Carroll's Tree Diagram for ARGUMENT 2

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