HOMEWORK 3 SOLUTIONS - MATH 300
INSTRUCTOR: George Voutsadakis

Problem 1 *The questions of Problem 1 refer to the following combined truth table:*

Line	P	Q	R	S	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
1	1	1	1	1	0	1	0	1	0	0	1	0	0	1	1	1
2	1	1	1	0	0	1	1	1	1	1	1	0	0	1	1	1
3	1	1	0	1	0	0	0	1	0	0	0	0	0	1	1	0
4	1	1	0	0	0	0	1	1	1	1	0	0	0	1	1	0
5	1	0	1	1	1	0	0	0	0	1	0	1	0	1	1	0
6	1	0	1	0	1	0	1	0	0	1	0	1	0	1	1	0
7	1	0	0	1	1	0	0	1	0	1	0	1	0	1	1	0
8	1	0	0	0	1	0	1	1	0	1	0	0	0	1	1	0
9	0	1	1	1	0	0	1	0	1	0	0	0	0	0	1	0
10	0	1	1	0	0	0	0	0	0	1	0	0	0	1	1	0
11	0	1	0	1	0	0	1	1	1	0	0	0	0	0	1	0
12	0	1	0	0	0	0	1	1	0	1	0	0	0	1	1	0
13	0	0	1	1	1	0	1	0	1	1	0	0	0	1	1	0
14	0	0	1	0	0	0	0	0	0	1	0	0	0	1	1	0
15	0	0	0	1	0	0	1	1	0	1	0	0	0	0	1	0
16	0	0	0	0	0	0	1	1	0	1	0	0	0	1	1	0

(a) *Which of the formulas F_1-F_{12} are truth equivalent?*

(b) *Which of the formulas F_1-F_{12} are tautologies?*

(c) *Which of the formulas F_1-F_{12} are contradictions?*

(d) *Determine if the following arguments are valid. If not, cite the number of a row of the truth table that refutes the argument.*

  (i) $F_1, F_5, F_{10} \therefore F_8$

  (ii) $F_1, F_3, F_4, F_5, F_6, F_7, F_{10} \therefore F_8$

(e) *Determine if the following collections of formulas are satisfiable. If so, cite the number of a row of the truth table that satisfies them.*

  (i) $\{F_1, F_5, F_6\}$

  (ii) $\{F_3, F_4, F_6, F_8\}$

(f) *Find the disjunctive normal form of F_1 (with respect to the variables P, Q, R, S).*

(g) *Find the conjunctive normal form of F_{10} (with respect to the variables P, Q, R, S).*

Solution:

(a) The only formulas that have identical truth tables are F_2, F_7 and F_{12}. So these are the only truth equivalent formulas among the ones given.

(b) Formula F_{11} is the only tautology.

(c) Formula F_9 is the only contradiction.

(d) The argument $F_1, F_5, F_{10} \therefore F_8$ is **not** valid. The assignment in Line 13 of the table makes all three hypotheses true and the conclusion false.

On the other hand the argument $F_1, F_3, F_4, F_5, F_6, F_7, F_{10} \therefore F_8$ is a valid argument since the set $\{F_1, F_3, F_4, F_5\}$ is unsatisfiable.

(e) The collection $\{F_1, F_5, F_6\}$ is satisfiable, as witnessed by the assignment of Line 13 .

On the other hand, the collection $\{F_3, F_4, F_6, F_8\}$ is unsatisfiable.

(f) The disjunctive normal form of F_1 is

$$(P \wedge \neg Q \wedge R \wedge S) \vee (P \wedge \neg Q \wedge R \wedge \neg S) \vee (P \wedge \neg Q \wedge \neg R \wedge S) \vee (P \wedge \neg Q \wedge \neg R \wedge \neg S) \vee (\neg P \wedge \neg Q \wedge R \wedge S).$$

(g) The conjunctive normal form of F_{10} is

$$(P \vee Q \vee R \vee \neg S) \wedge (P \vee \neg Q \vee R \vee \neg S) \wedge (P \vee \neg Q \vee \neg R \vee \neg S).$$

■

Problem 2 *Consider the following propositions:*

A	*"is able to stir the hearts of men"*
C	*"is clever"*
S	*"is Shakespeare"*
P	*"is a true poet"*
N	*"understands human nature"*
H	*"is the writer of Hamlet".*

Express the following as an argument in the propositional calculus and determine if it is valid, where the universe of discourse is the class of all writers:

All writers who understand human nature are clever.
No writer is a true poet unless he can stir the hearts of men.
Shakespeare wrote Hamlet.
No writer who does not understand human nature can stir the hearts of men.
None but a true poet could have written Hamlet.
$\therefore$ *Shakespeare is clever.*

Solution: Taking into account the variables denoting the corresponding propositions, we translate the given statements in the boxed argument as follows:

All writers who understand human nature are clever.	$N \to C$
No writer is a true poet unless he can stir the hearts of men.	$P \to A$
Shakespeare wrote Hamlet.	$S \to H$
No writer who does not understand human nature can stir the hearts of men.	$A \to N$
None but a true poet could have written Hamlet.	$H \to P$
$\therefore$ Shakespeare is clever.	$\therefore\ S \to C$

Assume now that all hypotheses are true. Then, if S is assigned the value true, by the 3rd statement, H is also assigned the value true. Thus, by the 5th statement, P is also assigned the value true. Hence, by the 2nd statement A is also assigned the value true. Thus, by the 4th statement N is assigned the value true as well and, therefore, by the 1st statement, C is also assigned the value true. This shows that $S \to C$ must be evaluated to true, provided that all hypotheses are evaluated to true. Therefore, the given argument is indeed a valid argument. ■

Problem 3 *Determine if the following argument is a valid argument:*

$$\neg C \wedge D$$
$$\neg(\neg B \wedge C \wedge D)$$
$$\neg(\neg B \vee (\neg A \wedge B)) \wedge \neg C \wedge \neg D$$
$$\therefore\ A \wedge \neg B.$$

Solution: The argument is vacuously valid: If D is assigned the truth value true, then the 3rd hypothesis is evaluated to false. On the other hand, if D is assigned the truth value false, then the 1st hypothesis is evaluated to false. Thus, the set consisting of the three hypotheses is an unsatisfiable set of formulas. ■

Problem 4 *Let $\mathcal{S}$ be an arbitrary infinite set of natural numbers, presented in binary notation (e.g., 12 is presented as* 1100*). Prove that there is an infinite sequence of different binary numbers $b_1, b_2, \ldots$, such that each b_i is a prefix of b_{i+1} and also a prefix of some element of $\mathcal{S}$.*

Solution: Suppose that the set $\mathcal{S}$ consists of the following natural numbers, thinking of them as written in binary notation:

$$s_1, s_2, s_3, \ldots, s_j, \ldots$$

For every $i = 1, 2, \ldots$, there are 2^{i-1} different binary strings of length i starting with digit 1. At least one and as many as 2^{i-1} of those strings appear as prefixes of strings in $\mathcal{S}$. Let $k_i \geq 1$ be the smallest index, such that in the finite sequence $s_1, s_2, \ldots, s_{k_i}$ all the different prefixes of length i appearing in the sequence $s_1, s_2, \ldots$ are present.

We will show, using compactness, that there exists an infinite sequence of different binary numbers $b_1, b_2, \ldots, b_i, \ldots$, such that

- b_i has i binary digits;
- each b_i is a prefix of b_{i+1};
- each b_i is a prefix of some element of $\mathcal{S}$.

To prove this, we introduce an infinite doubly-indexed collection of propositional variables $P_{i,j}$, $i, j = 1, 2, \ldots$. The intended meaning is that $P_{i,j}$ is going to be assigned value 1 if b_i is a prefix of s_j and value 0, otherwise. Then, we form the infinite set $\mathcal{X}$ of propositional formulas using the propositional variables $P_{i,j}, i, j = 1, 2, \ldots$ by adding the following classes of formulas in $\mathcal{X}$:

1. $P_{i,j} \to P_{i-1,j}$, for all $i = 2, 3, \ldots, j = 1, 2, \ldots$; Formulas of this type mean that if b_i is a prefix of s_j, then b_{i-1} will also be a prefix of s_j;

2. $\bigvee\{P_{i,j} : 1 \leq j \leq k_i$ and s_j has length at least $i\}$, for all $i = 1, 2, \ldots$; Formulas of this type mean that b_i is a prefix of at least one number from $s_1, \ldots, s_{k_i}$ in $\mathcal{S}$ (recall these are our only possible choices for strings of length i);

3. $\neg P_{i,p} \vee \neg P_{i,q}$, for all $i = 1, 2, \ldots$ and all $1 \leq p < q \leq k_i$, such that s_p and s_q do not have the same prefix of length i; Formulas of this type mean that b_i cannot be the prefix of two strings that do not share the same prefix of length i.

Once we have settled on this set $\mathcal{X}$ of propositional formulas, we must show that it serves our purposes. Namely, the following must be accomplished:

(a) It must be shown that every finite subset $\mathcal{Y}$ of $\mathcal{X}$ is satisfiable. Since every subset of a satisfiable set of formulas is also satisfiable, it does not harm generality to assume that $\mathcal{Y}$ consists of all formulas involving variables $P_{i,j}$, with $i \leq m, j \leq k_i$, for some fixed natural number m.

(b) By appealing to compactness, the previous proof will imply that $\mathcal{X}$ is satisfiable.

(c) The last step involves showing that the satisfiability of (the entire infinite set) $\mathcal{X}$ implies the existence of a sequence $b_1, b_2, \ldots$ of binary numbers that satisfy the requirements of the problem.

Proof of Item (a): Let m be the largest i, such that $P_{i,j}$ appears in the finite set $\mathcal{Y}$. Take b_m to be the prefix of s_{k_m} (note this determines also all b_i, for $i < m$, as the prefixes of length i of b_m) and define $P_{i,j}$ to be 1 if b_i is a prefix of s_j of length i and 0, otherwise. By the construction of the b_i's, for $i \leq m$, the set of formulas of type 1 in $\mathcal{Y}$ are satisfied. Formulas of type 2 are also satisfied, because, if b_m is a prefix of the formula s_{k_m}, then b_i, for $i < m$ is a prefix of length i of some formula s_j, with $j \leq k_m$. Finally, formulas of type 3 are also satisfied by the assignment of truth values to $P_{i,j}$. This finishes the proof of Item (a).
Proof of Item (c): We have now concluded that $\mathcal{X}$ is satisfiable. Assume that we have a satisfying assignment $\mathbf{e}$ of truth values to the variables $P_{i,j}$. We will informally refer to $P_{i,j}$ as been true or false depending on the truth value that it has been assigned under the assignment $\mathbf{e}$, satisfying $\mathcal{X}$. By the set of propositions of type 2, for every i, at least one of $P_{i,j}$ with $1 \leq j \leq k_i$ and s_j of length at least i, must be true. Moreover, by the propositions of type 2, if two of these $P_{i,j}$'s are true, then they agree on prefixes of length i. Let b_i be the unique prefix of the s_j's of length at least i for which $P_{i,j}$ is true. Then, for every i, b_i is a prefix of b_{i+1} by the propositions of type 1, and every b_i is the prefix of some number in $\mathcal{S}$, by formulas of type 2. ■

Problem 5 (*Generalization of the Erdős - De Bruijn Theorem*) *In this problem, the following definition is needed:*

> *A* **homomorphism** *f from a graph* **G** *to a graph* **H** *is a map $f : G \to H$, such that, if a and b are adjacent in* **G**, *then $f(a)$ and $f(b)$ are adjacent in* **H**, *i.e., such that f* **preserves the adjacency relation**, *or f* **preserves edges**.

If **H** *is a finite graph, prove that there is a homomorphism from* **G** *to* **H** *iff for every finite subgraph* $\mathbf{G}_0$ *of* **G**, *there is a homomorphism from* $\mathbf{G}_0$ *to* **H**.

(**Caution:** *In* graph theory, *what we call here a subgraph is usually referred to as an* **induced subgraph**; *In* logic *the usage accords with the concept of a* **substructure** *of an arbitrary first-order structure.*)

Solution: We use compactness to prove the statement. For all $a \in G$ and all $x \in H$, we introduce a variable P_{ax}. The intuition is that P_{ax} will have value 1 if vertex $a \in G$ gets mapped to vertex $x \in H$ and 0, otherwise. Let $\mathcal{S}$ be the set of propositional formulas consisting of

1. $\bigvee\{P_{ax} : x \in H\}$, $a \in G$; Formulas of this type mean that each $a \in G$ must be mapped to some vertex in H;

2. $\neg P_{ax} \vee \neg P_{ay}$, $a \in G, x, y \in H, x \neq y$; Formulas of this type ensure that each $a \in G$ is mapped to a unique vertex in H;

3. $\bigvee\{(P_{ax} \wedge P_{by}) : (x, y)$ an edge in $H\}$, for a every edge $(a, b) \in G$; Formulas of this type ensure that each pair (a, b) of vertices joined by an edge in G must be mapped to a pair (x, y) of vertices in H that are joined by an edge.

The hypothesis asserts the existence of a homomorphism for every finite subgraph $\mathbf{G}_0$ of **G** to **H**. This implies that every finite subset of $\mathcal{S}$ is satisfiable. By the Compactness Theorem (the entire infinite set of formulas) $\mathcal{S}$ is satisfiable. A satisfying assignment of truth values to the P_{ax}'s translates directly to a graph homomorphism from **G** to **H**. ■