HOMEWORK 4 - MATH 300 Due: Monday, March 18

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Read each problem **very carefully** and try to understand it well before starting to solve it. It is necessary to show **all** your work. Correct answers without explanations are worth 0 points. Write your own solutions and be neat!! **Take pride in your work!! Do not hand in scratchy doodles.**

1. Consider the language of algebras $\mathcal{L}$ consisting of the set of function symbols $\mathcal{F} = \{\bowtie, \rhd\}$, where $\bowtie$ is binary and $\rhd$ is unary, and the set of constants $\mathcal{C} = \emptyset$. Provide a list of all possible $\mathcal{L}$-terms in prefix notation (without using parentheses) that contain only the variable $x \in X$ and are of length at most 5 (i.e., contain at most five symbols).

2. How many $\mathcal{L}$-structures can one find on the set $A = \{a, b\}$ when $\mathcal{L} = \{+, \cdot, -, 0, 1\}$, with $+, \cdot$ binary function symbols, $-$ a unary function symbol and $0, 1$ constant symbols?
(**Hint:** See Exercise 3.1.4 of our textbook.)

3. Note that

   > A binary operation $g : A^2 \to A$ on a set A is **idempotent** if $g(a, a) = a$, for all $a \in A$.
   > An element $a \in A$ is a **fixed-point** of a unary operation $f : A \to A$ if $f(a) = a$.

   Find the number of $\mathcal{L}$-structures on the set $A = \{a, b, c\}$ when $\mathcal{L} = \{\vee, \wedge, ', 0, 1\}$, with $\vee, \wedge$ binary and $'$ a unary function symbol and $0, 1$ constant symbols, if one insists that

   - $\vee$ and $\wedge$ are interpreted as idempotent operations in A;
   - no element of A is a fixed-point under the interpretation of $'$ in A.

4. (**Syntax**) Consider the language of algebras $\mathcal{L}$ consisting of the set of function symbols $\mathcal{F} = \{f, g, h\}$, with f unary, g binary and h ternary, and the set of constant symbols $\mathcal{C} = \emptyset$. Consider, also
$$t(x, y, z) = gfgxzhygzfxfy.$$
   (a) Show the run on t of our γ-algorithm for determining syntactic validity and state clearly the conclusions of the algorithm.
   (b) Create the syntax tree for t.
   (c) Use the recursive definition of subterms to find all subterms of t; show carefully how each step of the recursive procedure is applied in finding the set of all subterms.

5. (**Semantics**) Consider the language of algebras $\mathcal{L}$ consisting of the set of function symbols $\mathcal{F} = \{f, g, h\}$, with f unary, g binary and h ternary, and the set of constant symbols $\mathcal{C} = \emptyset$. Moreover, let $A = \{a, b, c\}$ and interpret the three functions symbols as the functions $f^{\mathbf{A}} : A \to A$, $g^{\mathbf{A}} : A^2 \to A$ and $h^{\mathbf{A}} : A^3 \to A$, given by the following:

	$f^{\mathbf{A}}$
a	b
b	c
c	a

$g^{\mathbf{A}}$	a	b	c
a	a	b	c
b	c	c	a
c	c	b	b

$$h^{\mathbf{A}}(x, y, z) = \begin{cases} w, & \text{if at least two of } x, y, z \text{ are equal to } w \\ y, & \text{otherwise} \end{cases}$$

   (a) Consider again the term $t(x, y, z) = gfgxzhygzfxfy$ and compute the values
      (i) $t^{\mathbf{A}}(a, a, c)$;

(ii) $t^{\mathbf{A}}(a,b,c)$;
(iii) $t^{\mathbf{A}}(b,a,b)$;
(iv) $t^{\mathbf{A}}(b,c,a)$;
(v) $t^{\mathbf{A}}(c,c,c)$.

(b) Create the full Cayley table for $s^{\mathbf{A}}(x,y)$, where $s(x,y) = gxfgyx$.

6. Suppose that our language of algebras has a ternary function symbol h. Moreover, let $A = \{0,1,2\}$ and suppose that $h^{\mathbf{A}} : A^3 \to A$ is the function

$$h^{\mathbf{A}}(a,b,c) = a - b + c \ (\mathrm{mod}\ 3).$$

Find the values of the term functions associated with the indicated term at the arguments shown:

(a) $t(x) = hhxxxxhxxx$ at 1;
(b) $t(x,y) = hxhxyxy$ at $(0,1)$;
(c) $t(x,y,z) = hxhyzhxyzz$ at $(0,1,0)$;
(d) $t(x,y,z,w) = hhyzhwxyzhwxw$ at $(0,1,0,1)$.