HOMEWORK 6 SOLUTIONS - MATH 300
INSTRUCTOR: George Voutsadakis

Problem 1 *Consider the first-order language* $\mathcal{L} = \{f, r\}$, *where* f *is a ternary function symbol and* r *a binary relation symbol. Consider also the* $\mathcal{L}$*-structure* $\mathbf{S} = (S, I)$, *where* $S = \{0, 1\}$ *and* I *consists of the following interpretations of the function symbol* f *and the relation symbol* r*:*

			$f^{\mathbf{S}}$
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

$r^{\mathbf{S}}$	0	1
0	0	1
1	0	0

Determine the truth of the following first-order sentences in $\mathbf{S}$*:*

(a) $\forall x(rxfxxx)$

(b) $\forall x\exists y(rfxyxfxxx)$

Solution:

(a) The sentence $\forall x(rxfxxx)$ is false in the structure $\mathbf{S}$, since $r^{\mathbf{S}}(1, f^{\mathbf{S}}111)$ is equivalent to $r^{\mathbf{S}}(1, 0)$, which does not hold in $\mathbf{S}$.

(b) The sentence $\forall x\exists y(rfxyxfxxx)$ is false in the structure $\mathbf{S}$, since, for $x = 1$, we have that both $r^{\mathbf{S}}(f^{\mathbf{S}}101, f^{\mathbf{S}}111)$ and $r^{\mathbf{S}}(f^{\mathbf{S}}111, f^{\mathbf{S}}111)$, which are equivalent to $r^{\mathbf{S}}(1, 0)$ and $r^{\mathbf{S}}(0, 0)$, respectively, are false in $\mathbf{S}$.

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Problem 2 *Consider the first-order language* $\mathcal{L} = \{f, r\}$, *where* f *is a ternary function symbol and* r *a binary relation symbol. Consider also the* $\mathcal{L}$*-structure* $\mathbf{S} = (S, I)$, *where* $S = \{0, 1, 2\}$ *and* I *consists of the following interpretations of the function symbol* f *and the relation symbol* r*:*

$$f^{\mathbf{S}}(a, b, c) = (a - b + c) \bmod 3$$

$r^{\mathbf{S}}$	0	1	2
0	1	0	0
1	1	1	1
2	1	0	1

Determine the truth of the following first-order sentences in $\mathbf{S}$*:*

(a) $\forall x\exists y((rfxyxy) \to (ryfxyx))$

(b) $\forall x\exists y\forall z(rxfzyx)$

Solution:

(a) The sentence $\forall x\exists y((rfxyxy) \to (ryfxyx))$ is true in $\mathbf{S}$. In fact, for any $x = a \in S$, we may choose $y = a$, so that $r^{\mathbf{S}}(f^{\mathbf{S}}aaa, a) \to r^{\mathbf{S}}(a, f^{\mathbf{S}}aaa)$, which is equivalent to $r^{\mathbf{S}}(a, a) \to r^{\mathbf{S}}(a, a)$, is true in $\mathbf{S}$;

(b) The sentence $\forall x \exists y \forall z (rxfzyx)$ is false in $\mathbf{S}$; we argue by contradiction. If the sentence is true in $\mathbf{S}$, then it must hold for $x = 0$. Thus, there must exist some $y = b \in S$, such that, for all $z \in S$, $r^{\mathbf{S}}(0, f^{\mathbf{S}} zb0)$ is true in $\mathbf{S}$. Now we argue by cases depending on the value of $b \in S$ that this is impossible:

- If $b = 0$, then $r^{\mathbf{S}}(0, f^{\mathbf{S}}100)$ (equivalent to $r^{\mathbf{S}}(0, 1)$) is false;
- If $b = 1$, then $r^{\mathbf{S}}(0, f^{\mathbf{S}}210)$ (equivalent to $r^{\mathbf{S}}(0, 1)$) is false;
- If $b = 2$, then $r^{\mathbf{S}}(0, f^{\mathbf{S}}120)$ (equivalent to $r^{\mathbf{S}}(0, 2)$) is false.

So, for $x = 0$, there does not exist any $y \in S$, such that, for all $z \in S$, $rxfzyx$ is true in $\mathbf{S}$. ■

Problem 3 *Consider the language $\mathcal{L} = \{+, \cdot, <\}$, where $+, \cdot$ are binary function symbols and $<$ is a binary relation symbol. Let $x \leq y$ be an abbreviation for $(x < y) \vee (x \approx y)$. Use the skeleton method to determine whether each of the following $\mathcal{L}$-sentences has a one-element model. If yes, then exhibit that one-element model.*

(a) $\forall x (\exists y (x \cdot y < x) \to \exists z ((y \cdot z < x) \wedge \neg (y \cdot x < z))) \to \exists x \forall y (x \cdot y < y)$

(b) $\forall x (\exists y (x \cdot y \leq x) \to \neg \exists y \forall z ((y + z < x \cdot y) \wedge \neg (y \cdot x \leq z))) \to \exists x \forall y (x + y < x \cdot y)$

Solution:

(a) Consider first

$$\forall x (\exists y (x \cdot y < x) \to \exists z ((y \cdot z < x) \wedge \neg (y \cdot x < z))) \to \exists x \forall y (x \cdot y < y).$$

Start by removing all quantifiers and all terms:

$$((<) \to ((<) \wedge \neg (<))) \to (<).$$

Now replace $<$ by a propositional letter P:

$$(P \to (P \wedge \neg P)) \to P.$$

This last propositional formula is satisfiable when P is assigned the truth value 1. Therefore, the following $\mathcal{L}$-structure is a one-element model for $\forall x (\exists y (x \cdot y < x) \to \exists z ((y \cdot z < x) \wedge \neg (y \cdot x < z))) \to \exists x \forall y (x \cdot y < y)$:

$$S = \{a\}, \quad a +^{\mathbf{S}} a = a, \quad a \cdot^{\mathbf{S}} a = a, \quad <^{\mathbf{S}} = \{(a, a)\}.$$

(b) Next, consider

$$\forall x (\exists y (x \cdot y \leq x) \to \neg \exists y \forall z ((y + z < x \cdot y) \wedge \neg (y \cdot x \leq z))) \to \exists x \forall y (x + y < x \cdot y).$$

Expand the abbreviations $\leq$:

$$\forall x (\exists y ((x \cdot y < x) \vee (x \cdot y \approx x)) \to \neg \exists y \forall z ((y + z < x \cdot y) \wedge \neg ((y \cdot x < z) \vee (y \cdot x \approx z)))) \to \exists x \forall y (x + y < x \cdot y).$$

Remove all quantifiers and all terms:

$$(((<) \vee (\approx)) \to \neg ((<) \wedge \neg ((<) \vee (\approx)))) \to (<).$$

Finally, replace all $\approx$ by 1 and all $<$ by a propositional variable P:

$$((P \vee 1) \to \neg (P \wedge \neg (P \vee 1))) \to P.$$

This formula is equivalent to $1 \to P$, which is satisfiable when P is assigned the truth value 1. Therefore, the following $\mathcal{L}$-structure is a one-element model for $\forall x (\exists y (x \cdot y < x) \to \exists z ((y \cdot z < x) \wedge \neg (y \cdot x < z))) \to \exists x \forall y (x \cdot y < y)$:

$$S = \{a\}, \quad a +^{\mathbf{S}} a = a, \quad a \cdot^{\mathbf{S}} a = a, \quad <^{\mathbf{S}} = \{(a, a)\}.$$

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Problem 4 *To show that two $\mathcal{L}$-sentences F and G are equivalent, we must either use some of our fundamental equivalences from first-order logic to transform one to the other, or we have to prove that they are true in exactly the same structures, i.e., that for every structure $\mathbf{S}$, $\mathbf{S} \models F$ iff $\mathbf{S} \models G$. On the other hand, to show that they are not equivalent, it suffices to find a single structure $\mathbf{S}$ in which one of the two sentences is true and the other is false.*

Determine if the following pairs of $\mathcal{L}$-sentences are equivalent, where $\mathcal{L} = \{f\}$, with f a unary function symbol:

(a) $\forall x((ffx \approx x) \wedge (fffx \approx x))$ *and* $\forall x(fx \approx x)$;

(b) $\forall x \forall y((fx \approx fy) \rightarrow (x \approx y))$ *and* $\forall y \exists x(fx \approx y)$.

Solution:

(a) The sentences $\forall x((ffx \approx x) \wedge (fffx \approx x))$ and $\forall x(fx \approx x)$ are equivalent; To prove this, assume that $\mathbf{S} = (S, I)$ is an arbitrary $\mathcal{L}$-structure. We show that, for all $a \in S$,

$$(f^{\mathbf{S}} f^{\mathbf{S}} a = a \quad \text{and} \quad f^{\mathbf{S}} f^{\mathbf{S}} f^{\mathbf{S}} a = a) \quad \text{iff} \quad f^{\mathbf{S}} a = a.$$

The left-to-right implication is shown by

$$\begin{aligned} f^{\mathbf{S}} a &= f^{\mathbf{S}} f^{\mathbf{S}} f^{\mathbf{S}} a \quad \text{(by hypothesis)} \\ &= a \quad \text{(by hypothesis)} \end{aligned}$$

The right-to left-implication is shown by

$$f^{\mathbf{S}} f^{\mathbf{S}} a = f^{\mathbf{S}} a = a \quad \text{(both equations by hypothesis)}$$

and

$$f^{\mathbf{S}} f^{\mathbf{S}} f^{\mathbf{S}} a = f^{\mathbf{S}} f^{\mathbf{S}} a = f^{\mathbf{S}} a = a \quad \text{(all equations by hypothesis).}$$

(b) The sentences $\forall x \forall y((fx \approx fy) \rightarrow (x \approx y))$ and $\forall y \exists x(fx \approx y)$ are not equivalent; To show this consider the structure $\mathbf{S} = (S, f^{\mathbf{S}})$, where $S = \mathbb{N} = \{0, 1, 2, \ldots\}$ and $f^{\mathbf{S}}(n) = n + 1$, for all $n \in \mathbb{N}$.

Clearly, for all $n, m \in \mathbb{N}$, if $f^{\mathbf{S}}(n) = f^{\mathbf{S}}(m)$, then $n + 1 = m + 1$, whence $n = m$. This shows that $\forall x \forall y((fx \approx fy) \rightarrow (x \approx y))$ is true in $\mathbf{S}$.

On the other hand, for $y = 0$, there does not exist $n \in \mathbb{N}$, such that $0 = n + 1 = f^{\mathbf{S}}(n)$. Thus, $\forall y \exists x(fx \approx y)$ is not true in $\mathbf{S}$. Therefore, $\forall x \forall y((fx \approx fy) \rightarrow (x \approx y))$ and $\forall y \exists x(fx \approx y)$ are not true in exactly the same first-order structures and, hence, are not equivalent.

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Problem 5 *To show that two $\mathcal{L}$-formulas $F(x)$ and $G(x)$, with a free variable x are equivalent, we must either use some of our fundamental equivalences from first-order logic to transform one to the other, or we have to prove that they define exactly the same unary relations in all $\mathcal{L}$-structures, i.e., that for every structure $\mathbf{S}$ and every $a \in S$, $F^{\mathbf{S}}(a)$ holds iff $G^{\mathbf{S}}(a)$ holds. On the other hand, to show that they are not equivalent, it suffices to find a single structure $\mathbf{S}$ and a single element $a \in S$, such that one of $F^{\mathbf{S}}(a), G^{\mathbf{S}}(a)$ is true and the other is false.*

Determine if the following pairs of formulas are equivalent:

(a) $\forall y(rxy)$ *and* $\exists y(rxy)$, *where* $\mathcal{L} = \{r\}$, *r a binary relation symbol;*

(b) $\exists y(r_1 fy \wedge r_2 y \wedge (x \approx fy))$ *and* $\exists y \exists z(r_1 y \wedge r_2 z \wedge (x \approx fy) \wedge (x \approx fz))$, *where* $\mathcal{L} = \{f, r_1, r_2\}$, *with f a unary function symbol and r_1, r_2 unary relation symbols.*

Solution:

(a) The formulas $F(x) = \forall y(rxy)$ and $G(x) = \exists y(rxy)$ are not equivalent: Consider the first-order $\mathcal{L}$-structure $\mathbf{S} = (S, I)$, such that

$$S = \{a, b\}, \quad r^{\mathbf{S}} = \{(a, a)\}.$$

Then, since there does not exist any $x \in S$, such that, for all y, rxy holds in $\mathbf{S}$, we have $F^{\mathbf{S}} = \emptyset$. On the other hand, since $r^{\mathbf{S}}(a, a)$ holds, $x = a$ satisfies $G(x)$, whence $G^{\mathbf{S}} = \{a\}$. Since in $\mathbf{S}$, $F^{\mathbf{S}} \neq G^{\mathbf{S}}$, $F(x)$ and $G(x)$ are not equivalent formulas.

(b) The formulas $F(x) = \exists y(r_1 fy \wedge r_2 y \wedge (x \approx fy))$ and $G(x) = \exists y \exists z(r_1 y \wedge r_2 z \wedge (x \approx fy) \wedge (x \approx fz))$ are not equivalent formulas. Consider the first-order $\mathcal{L}$-structure $\mathbf{S} = (S, I)$, such that

$$S = \{a, b\}, \quad \begin{array}{c|c} & f^{\mathbf{S}} \\ \hline a & b \\ b & a \end{array}, \quad r_1^{\mathbf{S}} = \{a\}, \quad r_2^{\mathbf{S}} = \{b\}.$$

We will show that $F^{\mathbf{S}}(a)$ holds, whereas $G^{\mathbf{S}}(a)$ does not hold.

Since $r_1^{\mathbf{S}}(f^{\mathbf{S}} b)$ and $r_2^{\mathbf{S}}(b)$ and $a = f^{\mathbf{S}} b$ all hold, we have that $\exists y(r_1 fy \wedge r_2 y \wedge (x \approx fy))$ holds at $x = a$ in $\mathbf{S}$ with witness $y = b$.

On the other hand, for $x = a$, since the only element y in $\mathbf{S}$, such that $f^{\mathbf{S}} y = a$ is b, for $G^{\mathbf{S}}(a)$ to hold, we must have that the witness for the variable y in $G(x)$ at $x = a$ must be $y = b$. But $r_1^{\mathbf{S}}(b)$ does not hold. Thus, there is no element y bor which both statements $r_1^{\mathbf{S}} y$, $a = f^{\mathbf{S}} y$ are true in $\mathbf{S}$. Hence, $G^{\mathbf{S}}(a)$ is not true.

Since $F^{\mathbf{S}}(a)$ holds but $G^{\mathbf{S}}(a)$ does not hold, $F(x)$ and $G(x)$ are not equivalent formulas. ■

Problem 6 *Put the following formulas in the language of graphs into prenex normal form:*

(a) $(\forall y(\neg(y \approx z) \vee \forall y(y \approx z))) \vee (ryz)$

(b) $(z \approx x) \vee ((\forall x(\neg \forall z(rxz))) \to (y \approx z))$

(c) $(x \approx y) \to \exists y(((y \approx z) \to (\exists z(y \approx z))) \wedge (y \approx z))$

Solution:

(a) Consider the formula $(\forall y(\neg(y \approx z) \vee \forall y(y \approx z))) \vee (ryz)$ and apply the following steps:

Move	Resulting Formula
Original Formula	$(\forall y(\neg(y \approx z) \vee \forall y(y \approx z))) \vee (ryz)$
Rename Inside y	$(\forall y(\neg(y \approx z) \vee \forall w(w \approx z))) \vee (ryz)$
Rename Remaining y	$(\forall u(\neg(u \approx z) \vee \forall w(w \approx z))) \vee (ryz)$
Expand Scope of $\forall u$	$\forall u((\neg(u \approx z) \vee \forall w(w \approx z)) \vee (ryz))$
Expand Scope of $\forall w$	$\forall u \forall w((\neg(u \approx z) \vee (w \approx z)) \vee (ryz))$;

The last formula is in prenex normal form.

(b) For the formula $(z \approx x) \vee ((\forall x(\neg \forall z(rxz))) \to (y \approx z))$ we have

Move	Resulting Formula
Original Formula	$(z \approx x) \vee ((\forall x(\neg \forall z(rxz))) \to (y \approx z))$
Rename Inside z	$(z \approx x) \vee ((\forall x(\neg \forall w(rxw))) \to (y \approx z))$
Rename Inside x	$(z \approx x) \vee ((\forall u(\neg \forall w(ruw))) \to (y \approx z))$
Replace $\to$	$(z \approx x) \vee (\neg(\forall u(\neg \forall w(ruw))) \vee (y \approx z))$
Pull $\forall u$	$(z \approx x) \vee (\exists u(\neg(\neg \forall w(ruw))) \vee (y \approx z))$
Double Negation	$(z \approx x) \vee (\exists u(\forall w(ruw)) \vee (y \approx z))$
Exapand Scope of $\exists u$	$\exists u((z \approx x) \vee (\forall w(ruw) \vee (y \approx z))$
Exapand Scope of $\forall w$	$\exists u \forall w((z \approx x) \vee ((ruw) \vee (y \approx z)))$

The last formula is in prenex normal form.

(c) Finally, consider $(x \approx y) \to \exists y(((y \approx z) \to (\exists z(y \approx z))) \wedge (y \approx z))$. We have

Move	Resulting Formula
Original Formula	$(x \approx y) \to \exists y(((y \approx z) \to (\exists z(y \approx z))) \wedge (y \approx z))$
Rename Inside z	$(x \approx y) \to \exists y(((y \approx z) \to (\exists w(y \approx w))) \wedge (y \approx z))$
Rename Inside y	$(x \approx y) \to \exists u(((u \approx z) \to (\exists w(u \approx w))) \wedge (u \approx z))$
Replace the $\to$s	$\neg(x \approx y) \vee \exists u((\neg(u \approx z) \vee (\exists w(u \approx w))) \wedge (u \approx z))$
Expand Scope of $\exists w$ Twice	$\neg(x \approx y) \vee \exists u \exists w((\neg(u \approx z) \vee (u \approx w)) \wedge (u \approx z))$
Expand Scope of $\exists u$ and of $\exists w$	$\exists u \exists w(\neg(x \approx y) \vee ((\neg(u \approx z) \vee (u \approx w)) \wedge (u \approx z)))$

The last formula is in prenex normal form.

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Problem 7 *Find a counterexample for the following arguments:*

(a) *(Here* $\mathcal{L} = \{r_1, r_2, r_3\}$*, with* r_1, r_2, r_3 *unary relation symbols.)*

$$\forall x(r_1 x \to (r_2 x \to r_3 x))$$
$$\therefore\ \forall x((r_1 x \to r_2 x) \to r_3 x)$$

(b) *(Here* $\mathcal{L} = \{r\}$*, with* r *a binary relation symbol.)*

$$\forall x \exists y(rxy)$$
$$\forall y \exists x(rxy)$$
$$\therefore\ \forall x \forall y(\neg(x \approx y) \to (rxy))$$

Solution:

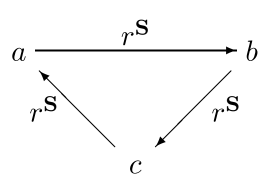
(a) Consider the $\mathcal{L}$-structure $\mathbf{S} = (S, I)$, where

$$S = \{a\}, \quad r_1^{\mathbf{S}} = \emptyset, \quad r_2^{\mathbf{S}} = \{a\}, \quad r_3^{\mathbf{S}} = \emptyset.$$

Then, it is true in $\mathbf{S}$ that $\forall x(r_1 x \to (r_2 x \to r_3 x))$, since $r_1^{\mathbf{S}} a \to (r_2^{\mathbf{S}} a \to r_3^{\mathbf{S}} a)$ is a true statement. On the other hand, it is false in $\mathbf{S}$ that $\forall x((r_1 x \to r_2 x) \to r_3 x)$, since $(r_1^{\mathbf{S}} a \to r_2^{\mathbf{S}} a) \to r_3^{\mathbf{S}} a$ is a false statement.

(b) Consider the $\mathcal{L}$-structure $\mathbf{S} = (S, I)$, such that

$$S = \{a, b, c\}, \quad r^{\mathbf{S}} = \{(a, b), (b, c), (c, a)\}.$$



Then, clearly, both $\forall x \exists y(rxy)$ (saying that "for every vertex, there exists an outgoing edge") and $\forall y \exists x(rxy)$ (saying that "for every vertex, there exists an incoming edge") are true in $\mathbf{S}$, whereas $\forall x \forall y(\neg(x \approx y) \to (rxy))$ is not true in $\mathbf{S}$, as, for instance $b \neq a \to r^{\mathbf{S}}(b, a)$ is a false statement. Therefore, the given argument is not a valid argument.

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Problem 8 *Skolemize the following sentences:*

(a) $\forall x \exists y(x < y)$

(b) $\exists x \forall y(x < y)$

(c) $\forall x \forall y ((rxy) \rightarrow \exists z ((rxz) \wedge (rzy)))$

Solution: In this solution, we write $\mathsf{Sk}(F)$ to denote the Skolemization of a sentence F.

(a) To Skolemize $\forall x \exists y (x < y)$, which is already in prenex normal form, we introduce a fresh unary function symbol f to express the dependency of y on x:

$$\mathsf{Sk}(\forall x \exists y (x < y)) \equiv \forall x (x < f(x)).$$

(b) To Skolemize $\exists x \forall y (x < y)$, which is already in prenex normal form, we introduce a fresh constant symbol c as a witness to the existing x:

$$\mathsf{Sk}(\exists x \forall y (x < y)) \equiv \forall y (c < y).$$

(c) To Skolemize $\forall x \forall y ((rxy) \rightarrow \exists z ((rxz) \wedge (rzy)))$, we first convert it to prenex normal form

Move	Resulting Formula
Original Formula	$\forall x \forall y ((rxy) \rightarrow \exists z ((rxz) \wedge (rzy)))$
Replace the $\rightarrow$	$\forall x \forall y (\neg (rxy) \vee \exists z ((rxz) \wedge (rzy)))$
Expand Scope of $\exists z$	$\forall x \forall y \exists z (\neg (rxy) \vee ((rxz) \wedge (rzy)))$.

Next, we introduce a fresh binary function symbol g to express the dependency of z on x and y:

$$\mathsf{Sk}(\forall x \forall y ((rxy) \rightarrow \exists z ((rxz) \wedge (rzy)))) \equiv \forall x \forall y (\neg (rxy) \vee ((rxgxy) \wedge (rgxyy))).$$

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