EXAM 1 - PROBLEM 1 SOLUTION

1. Find the length of the parametric curve $x = \dfrac{t}{1+t}$, $y = \ln(1+t)$, $0 \le t \le 2$.

$$
\begin{aligned}
L &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^2 \sqrt{\left(\frac{1}{(1+t)^2}\right)^2 + \left(\frac{1}{1+t}\right)^2}\,dt \\
&= \int_0^2 \sqrt{\frac{1}{(1+t)^4} + \frac{1}{(1+t)^2}}\,dt \\
&= \int_0^2 \frac{\sqrt{1+(1+t)^2}}{(1+t)^2}\,dt \\
&\overset{u=1+t}{=} \int_1^3 \frac{\sqrt{1+u^2}}{u^2}\,du \\
&= \int_1^3 \frac{1+u^2}{u^2\sqrt{1+u^2}}\,du \\
&= \int_1^3 \frac{1}{u^2\sqrt{1+u^2}}\,du + \int_1^3 \frac{u^2}{u^2\sqrt{1+u^2}}\,du \\
&= \int_1^3 \frac{1}{u^2\sqrt{1+u^2}}\,du + \int_1^3 \frac{1}{\sqrt{1+u^2}}\,du.
\end{aligned}
$$

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\begin{aligned}
\int_1^3 \frac{1}{u^2\sqrt{1+u^2}}\,du &\overset{v=\frac{\sqrt{1+u^2}}{u}}{=} \int_{\sqrt{2}}^{\sqrt{10}/3} -dv \\
&= -v\Big|_{\sqrt{2}}^{\sqrt{10}/3} \\
&= \sqrt{2} - \frac{\sqrt{10}}{3}.
\end{aligned}
$$

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\begin{aligned}
\int_1^3 \frac{1}{\sqrt{1+u^2}}\,du &\overset{u=\tan\theta}{=} \int_{\pi/4}^{\tan^{-1}3} \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta \\
&= \int_{\pi/4}^{\tan^{-1}3} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta \\
&= \int_{\pi/4}^{\tan^{-1}3} \frac{\sec^2\theta}{\sec\theta}\,d\theta \\
&= \int_{\pi/4}^{\tan^{-1}3} \sec\theta\,d\theta \\
&= \int_{\pi/4}^{\tan^{-1}3} \frac{(\sec\theta+\tan\theta)\sec\theta}{\sec\theta+\tan\theta}\,d\theta \\
&= \int_{\pi/4}^{\tan^{-1}3} \frac{\tan\theta\sec\theta+\sec^2\theta}{\sec\theta+\tan\theta}\,d\theta \\
&\overset{z=\sec\theta+\tan\theta}{=} \int_{\sqrt{2}+1}^{\sqrt{10}+3} \frac{1}{z}\,dz \\
&= \ln z\Big|_{\sqrt{2}+1}^{\sqrt{10}+3} \\
&= \ln(\sqrt{10}+3) - \ln(\sqrt{2}+1).
\end{aligned}
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